The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2019–2020

Continuum Mechanics — **2 hours**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Write out the following expressions in three dimensions without using suffix notations:

$$\text{(a)}\ \delta_{ij}x_i y_j, \quad \text{(b)}\ t_i = \varepsilon_{ijk}x_j y_k, \quad \text{(c)}\ V = T_{ij}U_{ji}.$$

(6 marks)

(ii) A new coordinate system is obtained from the old one by rotating the coordinate axes about the x_1 by the angle θ.

(a) Show that the transformation matrix from the old to the new coordinates is given by

$$\hat{\mathbf{A}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}.$$

Draw the figure clearly showing the old and new coordinate axes.

(5 marks)

(b) The components of the tensor $\mathbf{T}$ in the old coordinates are given by

$$\hat{\mathbf{T}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & b & c \end{pmatrix}.$$

Find the matrix $\hat{\mathbf{T}}'$ of components of this tensor in the new coordinates.

[You can use without proof that the matrices of components of tensor $\mathbf{T}$ in the old and new coordinates are related by $\hat{\mathbf{T}}' = \hat{\mathbf{A}}\hat{\mathbf{T}}\hat{\mathbf{A}}^T$].

(7 marks)

 Question 1 continued on next page

1 (continued)

(c) You are given that $a > c$, $0 < \theta < \dfrac{\pi}{4}$ and

$$\tan 2\theta = \frac{2b}{a-c}.$$

Show that

$$T'_{ij} = 0 \ (i \neq j) \quad \text{and} \quad T'_{33} = \frac{1}{2}\left(a + c - \sqrt{(a-c)^2 + 4b^2}\right).$$

[Hint: Use the formulae $\sec^2 2\theta = 1 + \tan^2 2\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$]. ***(7 marks)***

2 (i) Write down the differential equation determining a particle trajectory. ***(2 marks)***

(ii) The velocity field of a planar motion is given by

$$\boldsymbol{v} = \left[-\frac{a^3 x_2 (x_1^2 + x_2^2)^{1/2}}{T(2a^2 + x_1^2 + x_2^2)^2} + \frac{t x_1 a^2}{T^2(a^2 + x_1^2 + x_2^2)} \right] \boldsymbol{e}_1 + \left[\frac{a^3 x_1 (x_1^2 + x_2^2)^{1/2}}{T(2a^2 + x_1^2 + x_2^2)^2} + \frac{t x_2 a^2}{T^2(a^2 + x_1^2 + x_2^2)} \right] \boldsymbol{e}_2, \qquad (*)$$

where a and T are positive constants, and $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ are the base vectors.

(a) Use the variable substitution

$$x_1 = r\cos\phi, \quad x_2 = r\sin\phi$$

to write down the differential equations determining a particle trajectory in the x_1x_2-plane in polar coordinates r, ϕ from the velocity field $(*)$. ***(11 marks)***

(b) Initially the particle is at distance a from the coordinate origin, that is, $r = a$ when $t = 0$. Determine the time when the particle distance from the origin is $2a$. ***(12 marks)***

3 A solid sphere of radius R is embedded in a continuous medium. The sphere creates a gravity field with acceleration due to gravity directed to the centre of the sphere and inversely proportional to the square of the distance from it, i.e. in spherical coordinates r, θ, φ with the origin at the centre of the sphere the body force is $\boldsymbol{b} = (-gR^2/r^2, 0, 0)$. You are also given that the stress tensor has the form $\mathbf{T} = -p\mathbf{I}$, where $\mathbf{I}$ is the unit tensor and p the pressure.

(i) Use the equilibrium equation $\nabla \cdot \mathbf{T} + \rho\boldsymbol{b} = 0$, where ρ is the density, to show that p is determined by the equation

$$\nabla p = \rho \boldsymbol{b}.$$

(3 marks)

(ii) Show that p is independent of θ and ϕ.
[You can use $\nabla p = \dfrac{\partial p}{\partial r}\boldsymbol{e}_r + \dfrac{1}{r}\dfrac{\partial p}{\partial \theta}\boldsymbol{e}_\theta + \dfrac{1}{r\sin\theta}\dfrac{\partial p}{\partial \phi}\boldsymbol{e}_\phi$, where $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, and $\boldsymbol{e}_\phi$ are the unit vectors in the r, θ, and ϕ-direction]. ***(3 marks)***

(iii) You are now given that the pressure is related to the density by $p = (a\rho)^n$ with $a > 0$ and $0 < n \le 1$, and $p = p_0 = \text{const}$ at $r = R$. Determine the dependence of p on r for $r > R$ separately for $n = 1$ and $n < 1$. In both cases calculate the limiting value of pressure, p_∞, as $r \to \infty$. ***(19 marks)***

4 (i) Using Euler's equation for incompressible homogeneous fluid written in the Gromeka-Lamb form,

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\nabla \times \boldsymbol{v}) \times \boldsymbol{v} = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}\|\boldsymbol{v}\|^2 + \varphi\right),$$

where φ is the body force potential, derive Bernoulli's integral for fluid stationary motion,

$$p + \frac{\rho}{2}\|\boldsymbol{v}\|^2 + \rho\varphi = \text{const}$$

along a streamline. ***(8 marks)***

(ii) A tank in the form of a cylinder of radius $R = 1$ m and height H is open from above and filled with water up to the top. There is a circular hole of radius $r = 1$ cm at the bottom of the tank, so that water leaks through this hole.

(a) Considering the flow as approximately stationary and water as an ideal incompressible fluid, show that Bernoulli's integral takes the form

$$v^2 = 2gh,$$

where $v = \|\boldsymbol{v}\|$, g is the acceleration due to gravity, and h is the water depth in the tank. ***(5 marks)***

(b) Show that h satisfies the equation

$$\frac{dh}{dt} = -\frac{r^2\sqrt{2gh}}{R^2}$$

(5 marks)

(c) You are given that, after the time interval $T = 30$ min., the level of water in the tank is $h_0 = 1$ m. Obtain the expression for h_0 in terms of R, r, h_0, and g. Then use this expression to calculate the numerical value of H. [You may take $g = 10\,\text{m}\,\text{s}^{-2}$.] ***(7 marks)***

5 (i) In linear elasticity the equilibrium of an isotropic material is described by the equation

$$(\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) + \mu\nabla^2\boldsymbol{u} + \rho\boldsymbol{b} = 0. \qquad (*)$$

You are given that, in cylindrical coordinates r, ϕ, z, the displacement vector $\boldsymbol{u}$ has only the ϕ-component and it is independent of ϕ, $\boldsymbol{u} = u(r, z)\boldsymbol{e}_\phi$, where $\boldsymbol{e}_\phi$ is the unit vector in the ϕ-direction. In addition, there is no body force, $\boldsymbol{b} = 0$. Show that, in this case, equation $(*)$ reduces to

$$\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial(ru)}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0. \qquad (\dagger)$$

[You can use without proof the formulae

$$\nabla^2\boldsymbol{u} = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla \times \nabla \times \boldsymbol{u}, \quad \nabla \cdot \boldsymbol{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} \quad \text{and}$$

$$\nabla \times \boldsymbol{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \phi} - \frac{\partial u_\phi}{\partial z}\right)\boldsymbol{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\boldsymbol{e}_\phi + \frac{1}{r}\left(\frac{\partial(ru_\phi)}{\partial r} - \frac{\partial u_r}{\partial \phi}\right)\boldsymbol{e}_z\Big].$$

(6 marks)

(ii) There is an elastic rod of radius R and length L. The rod axis coincides with the z-axis of cylindrical coordinates. Its ends are fixed at the planes $z = 0$ and $z = L$. Initially there are no stresses in the rod. Then the upper plane is rotated by a small angle α, so the displacement at $z = L$ is $\boldsymbol{u} = (0, \alpha r, 0)$. The lower plane does not move, so the displacement is zero at $z = 0$.

(a) Assuming that the displacement in the rod has the form $\boldsymbol{u} = rf(z)\boldsymbol{e}_\phi$ use equation $(\dagger)$ to determine the function $f(z)$. ***(5 marks)***

(b) Use the expression of the stress tensor in terms of the displacement,

$$\mathbf{T} = \lambda\mathbf{I}\nabla \cdot \boldsymbol{u} + \mu\left[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right],$$

where $\mathbf{I}$ is the unit tensor, to calculate the stress tensor in the rod. [You can use without proof that, for $\boldsymbol{u} = u(r, z)\boldsymbol{e}_\phi$,

$$\nabla\boldsymbol{u} = \frac{\partial u}{\partial r}\boldsymbol{e}_r\boldsymbol{e}_\phi - \frac{u}{r}\boldsymbol{e}_\phi\boldsymbol{e}_r + \frac{\partial u}{\partial z}\boldsymbol{e}_z\boldsymbol{e}_\phi\Big].$$

(5 marks)

(c) Use the obtained expression for $\mathbf{T}$ to calculate the surface traction at the upper edge of the rod, $z = L$. Then determine the moment of force applied to the upper plane that is needed to rotate it by the angle α. ***(9 marks)***

End of Question Paper