The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2019–20**

Introduction to Relativity **2 hours**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

1 (i) (a) State the two postulates of special relativity. ***(4 marks)***

(b) Define what is meant by an *inertial frame*.
Give **one** example of an (approximately) inertial frame, and **one** example of a non-inertial frame. ***(4 marks)***

(ii) Two inertial frames $R : (ct, x)$ and $\widetilde{R} : (c\widetilde{t}, \widetilde{x})$ are related by the *Lorentz transformation*

$$\begin{pmatrix} c\widetilde{t} \\ \widetilde{x} \end{pmatrix} = \gamma(v) \begin{pmatrix} 1 & -v/c \\ -v/c & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}, \qquad \gamma(v) = \left(1 - \frac{v^2}{c^2}\right)^{-1/2},$$

with $v > 0$.

(a) A rod is at rest in frame R. An observer, also at rest in R, measures the positions of the two ends simultaneously at the events $O : (0, 0)$ and $A : (0, L)$.
Show that the corresponding events $\widetilde{O}$ and $\widetilde{A}$ are **not** simultaneous in $\widetilde{R}$. In which order do the events occur in $\widetilde{R}$? ***(4 marks)***

(b) An observer at rest in frame $\widetilde{R}$ measures the two ends of the rod simultaneously at events $\widetilde{O} : (0, 0)$ and $\widetilde{B} : (0, d)$.
Show that $d = L/\gamma(v)$. ***(5 marks)***

(c) Draw a *spacetime diagram* showing the coordinate axes for the two frames R and $\widetilde{R}$. Mark the events O, A and B on the diagram. ***(4 marks)***

(iii) Briefly describe the phenomenon of *length contraction* in special relativity, with reference to your answers in part (ii)(a) and (ii)(b). ***(4 marks)***

2 Two inertial frames $R:(ct,x,y,z)$ and $\widetilde{R}:(c\widetilde{t},\widetilde{x},\widetilde{y},\widetilde{z})$ are related by the homogeneous Lorentz transformation

$$\begin{pmatrix} c\widetilde{t} \\ \widetilde{x} \\ \widetilde{y} \\ \widetilde{z} \end{pmatrix} = L \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \qquad \text{where} \quad L^T g L = g.$$

(i) (a) Write down the 4×4 matrix for the metric tensor g. ***(2 marks)***

(b) Show that $\det L = \pm 1$. ***(3 marks)***

(c) Show that L^T is also a Lorentz transformation. ***(4 marks)***

(ii) Let

$$M = \begin{pmatrix} \sqrt{2} & 1 & 0 & 0 \\ 1 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad N = \frac{1}{2}\begin{pmatrix} -3 & 0 & -\sqrt{5} & 0 \\ 0 & 2 & 0 & 0 \\ \sqrt{5} & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

(a) Show that M is a proper orthochronous Lorentz transformation. ***(6 marks)***

(b) Show that N is a Lorentz transformation.
Is it proper? Is it orthochronous? ***(7 marks)***

(c) Let $L = MN$. Answer either True or False for each statement below:

1. L is a Lorentz transformation.
2. L is orthochronous.
3. L is proper.

(3 marks)

3 Two particles A and B are moving in an inertial frame R in the positive-x and positive-y directions, respectively, with four-velocities

$$U = \gamma_u \begin{pmatrix} c \\ u \\ 0 \\ 0 \end{pmatrix}, \quad V = \gamma_v \begin{pmatrix} c \\ 0 \\ v \\ 0 \end{pmatrix}, \qquad \gamma_u = \left(1 - \frac{u^2}{c^2}\right)^{-1/2}, \quad \gamma_v = \left(1 - \frac{v^2}{c^2}\right)^{-1/2},$$

where the speeds u and v are constants and c is the speed of light.

Let $\widetilde{R}$ be the rest frame of particle A, in which $\widetilde{U} = (c, 0, 0, 0)^T$.

(i) (a) Define the Lorentz bracket.
Show that $g(U, U) = c^2$.
Classify U as timelike, spacelike or null. ***(6 marks)***

(ii) (a) Write down the standard Lorentz transformation L between the inertial frames R and $\widetilde{R}$, such that $\widetilde{U} = LU = (c, 0, 0, 0)^T$. ***(3 marks)***

(b) Show that in $\widetilde{R}$, the rest frame of A, particle B has the four-velocity

$$\widetilde{V} = \gamma_u \gamma_v \begin{pmatrix} c \\ -u \\ v/\gamma_u \\ 0 \end{pmatrix}.$$

(3 marks)

(c) The particle B has speed w in the rest frame of A.
By writing down an expression for $\widetilde{V}$ in terms of w, and by using part (ii)(b), show that

$$\gamma_w = \gamma_u \gamma_v$$

and

$$w^2 = u^2 + v^2 - \frac{u^2 v^2}{c^2}.$$

(6 marks)

(iii) Now let $u = v = c/2$.

(a) Calculate the speed w. ***(2 marks)***

(b) Calculate the angle θ between the trajectory of particle B and the $\widetilde{y}$-axis in frame $\widetilde{R}$. ***(5 marks)***

4 A particle moving in an inertial frame R has a position vector $X(\tau)$ and a four-velocity $V(\tau) = \dfrac{dX}{d\tau}$ given by

$$V(\tau) = (c\cosh\rho,\ c\sinh\rho,\ 0,\ 0)\,,$$

where the rapidity ρ is a function of proper time τ.

(i) (a) Show that the four-acceleration A is given by

$$A(\tau) = c\frac{d\rho}{d\tau}\,(\sinh\rho, \cosh\rho, 0, 0)\,.$$

(3 marks)

(b) In the instantaneous rest frame of the particle $\widetilde{R}$, the four-acceleration has components $\widetilde{A} = (0, a, 0, 0)$. Show that

$$\left(\frac{d\rho}{d\tau}\right)^2 = \frac{a^2}{c^2}.$$

(4 marks)

(c) The particle starts from rest at the origin of frame R, and then undergoes uniform acceleration $a > 0$ in the positive-x direction. Show that the rapidity is $\rho = \dfrac{a\tau}{c}$.
Hence show that the displacement vector $X(\tau)$ is

$$X(\tau) = \frac{c^2}{a}\,(\sinh\rho, \cosh\rho - 1, 0, 0)\,.$$

(6 marks)

(ii) Starting from rest at $\tau = 0$ at the origin of R, a spaceship undergoes uniform acceleration a until it reaches position $x = d$. It then undergoes uniform deceleration, such that it comes to a rest at $x = 2d$.

(a) Show that the time taken for this journey in R is

$$t = \frac{2d}{c}\sqrt{1 + \frac{2c^2}{ad}}.$$

(8 marks)

(b) Let $a = 9.81\,\mathrm{ms}^{-2}$, $d = 1$ light year and $c \approx 3 \times 10^8\,\mathrm{ms}^{-1}$. Calculate the journey time in years, as measured in the frame R. ***(4 marks)***

5 (i) Define the *rest mass* and *four-momentum* of a particle. ***(3 marks)***

(ii) (a) Three identical particles of rest mass m are in uniform motion in an inertial frame R with identical speeds u. They are moving in the positive x-direction, positive y-direction and negative x-direction, respectively. Show that the total four-momentum P in R is

$$P = m\,(3c\gamma(u), 0, u\gamma(u), 0)\,.$$

(4 marks)

(b) The three particles undergo a collision at a single point and fuse together. Assuming conservation of four-momentum, find the speed of the composite particle in R. ***(5 marks)***

(c) Show that the rest mass of the composite particle is

$$m\sqrt{\frac{9c^2 - u^2}{c^2 - u^2}}\,.$$

(7 marks)

(iii) A single photon has energy $E = 3 \times 10^{-19}$J and zero rest mass.

(a) Use $E^2 = p^2c^2 + m^2c^4$ with $c \approx 3 \times 10^8\,\mathrm{ms}^{-1}$ to calculate p, the magnitude of the single photon's momentum. ***(2 marks)***

(b) Suppose 10^{22} such photons are directly reflected by a 'solar sail' of mass 1g which is initially at rest. Estimate the resulting speed of the solar sail. ***(4 marks)***

End of Question Paper