The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn 2019**

Topics in Number Theory **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Please read the questions carefully. Your solutions should be written legibly and give enough details to make it clear how you arrived at your answers. Usage of calculators is not allowed.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
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1 ***(18 marks)***

(i) State when an element $\overline{a} \in \mathbb{Z}_n$ has a multiplicative inverse. ***(2 marks)***

(ii) State the Modular Binomial Theorem $\pmod p$. ***(2 marks)***

(iii) Define the Euler ϕ function. ***(2 marks)***

(iv) Let a be a primitive root modulo n. Give the necessary and sufficient condition on j for a^j to be a primitive root. ***(2 marks)***

(v) State Euler's criterion for quadratic residues modulo a prime number p. ***(2 marks)***

(vi) What does it mean for a positive integer to be a perfect number? ***(2 marks)***

(vii) Define a primitive Pythagorean triple. ***(2 marks)***

(viii) QUESTION ON GENERATING FUNCTIONS REMOVED AS THE TOPIC WAS NOT COVERED DUE TO STRIKE ACTION (was 2 marks)

(ix) Explain the notation $[a_0, a_1, ..., a_k]$. ***(2 marks)***

(x) Given a positive integer solution (x, y) of $x^2 - dy^2 = 1$, how does it relate to the continued fraction expansion of $\sqrt{d}$? ***(2 marks)***

2 ***(48 marks)***

(i) Solve the linear congruence $56x \equiv 8 \pmod{72}$. ***(6 marks)***

(ii) Determine the remainder when 1234^{1234} is divided by 91 (note that $91 = 7 \cdot 13$). ***(6 marks)***

(iii) Find the formula for $\sum_{d|n} d^2$ in terms of the prime factorization $n = p_1^{k_1} \cdots p_r^{k_r}$, in the simple form, using the sum of a geometric series when appropriate. ***(6 marks)***

(iv) Let n be a positive integer.

(a) Prove that $\phi(n) = n - 1$ if and only if n is a prime.

(b) Prove that for all $n > 2$, $\phi(n)$ is even. ***(6 marks)***

(v) Let p be a prime, and assume that $p \neq 2, 5$. Compute $\left(\frac{5}{p}\right)$ in terms of $p \pmod 5$. ***(6 marks)***

(vi) Find all Pythagorean triples of the form $(18, y, z)$. ***(6 marks)***

(vii) Express $\frac{\sqrt{10}}{3}$ as an infinite periodic continued fraction. ***(6 marks)***

(viii) Find a primitive root modulo 17, and describe all other primitive roots in terms of its powers. ***(6 marks)***

3 ***(14 marks)***

(i) If $n > 6$ is an even perfect number, prove that $n \equiv 4 \pmod 6$. ***(7 marks)***

(ii) Define the group operation on the set of solutions of Pell's equation

$$G_d = \{(x, y) \in \mathbb{Z}^2 : x^2 - dy^2 = 1\}$$

and show that the group axioms are satisfied. ***(7 marks)***

(iii) QUESTION ON GENERATING FUNCTIONS REMOVED AS THE TOPIC WAS NOT COVERED DUE TO STRIKE ACTION (was 7 marks)

4 *(11 marks)*

In this question we investigate quadratic residues modulo composite integers. Let n be a positive integer; we say that an integer a is a quadratic residue modulo n if a is coprime to n and $\overline{a}$ is a square in $\mathbb{Z}_n^*$.

Let $\delta(n)$ be the number of quadratic residues modulo n. For example $\delta(4) = 1$ since $\mathbb{Z}_4^* = \{\overline{1}, \overline{3}\}$, and among these two elements only $\overline{1}$ is a quadratic residue.

(a) Compute $\delta(15)$. *(4 marks)*

(b) Show that $\delta(n)$ is an arithmetic multiplicative function, and deduce a formula for $\delta(n)$ when n is a product of distinct primes $p_1 \cdots p_r$. *(7 marks)*

End of Question Paper