



The  
University  
Of  
Sheffield.

**MAS 331**

**SCHOOL OF MATHEMATICS AND STATISTICS**

**Autumn Semester 2019**

**Metric Spaces**

**2 hours 30 minutes**

*Attempt all the questions. The allocation of marks is shown in brackets.*

**Please leave this exam paper on your desk  
Do not remove it from the hall**

Registration number from U-Card (9 digits)  
to be completed by student

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1 (i) Define the concept of metric space. (5 marks)

(ii) Define  $d_3: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$d_3(\underline{x}, \underline{y}) = |x^1 - y^1| + 3|x^2 - y^2|,$$

where  $\underline{x} = (x^1, x^2)$ ,  $\underline{y} = (y^1, y^2)$ .

(a) Show that  $d_3$  is a metric on  $\mathbb{R}^2$ .

(b) Show that

$$d_1(\underline{x}, \underline{y}) \leq d_3(\underline{x}, \underline{y}) \leq 3d_1(\underline{x}, \underline{y})$$

for all  $\underline{x}, \underline{y} \in \mathbb{R}^2$ . Here

$$d_1(\underline{x}, \underline{y}) = |x^1 - y^1| + |x^2 - y^2|,$$

as usual.

(6 marks)

(iii) Let  $(X, d)$  be a metric space.

(a) Prove that, for all  $x, y, z \in X$ ,

$$|d(x, z) - d(y, z)| \leq d(x, y).$$

(b) Prove that, for all  $x, y, a, b \in X$ ,

$$|d(x, y) - d(a, b)| \leq d(x, a) + d(y, b)$$

(6 marks)

(iv) Define what it means for a sequence  $(x_n)$  in a metric space  $(X, d)$  to converge to an element  $x \in X$ . (2 marks)

(v) Let  $(x_n)$  and  $(y_n)$  be two sequences in a metric space  $(X, d)$ .

(a) Show that if  $x_n \rightarrow a$  and  $y_n \rightarrow a$  then  $d(x_n, y_n) \rightarrow 0$ .

(b) Show that if  $x_n \rightarrow a$  and  $y_n \rightarrow b$  then  $d(x_n, y_n) \rightarrow d(a, b)$ .

(6 marks)

- 2 (i) Let  $f: X \rightarrow Y$  be a function from a metric space  $(X, d_X)$  to a metric space  $(Y, d_Y)$ . Define what it means for  $f$  to be  $\varepsilon\delta$ -continuous at  $x \in X$ , and what it means for  $f$  to be sequentially-continuous at  $x \in X$ .

(6 marks)

- (ii) Now consider a continuous function  $h: [a, b] \rightarrow \mathbb{R}$  where  $[a, b]$  is a closed bounded interval in  $\mathbb{R}$ . Suppose that  $h(c) > 0$  for a certain  $c \in (a, b)$ . Show that there exists  $A > 0$  and  $\delta > 0$  such that

$$h(x) > \frac{1}{2}A \text{ for all } c - \delta < x < c + \delta.$$

(4 marks)

- (iii) Define a function  $d_1: C[a, b] \times C[a, b] \rightarrow \mathbb{R}$  by

$$d_1(f, g) = \int_a^b |f(x) - g(x)| dx.$$

Prove that  $d_1$  is a metric on  $C[a, b]$ .

(10 marks)

- (iv) In the metric space  $(C[0, 1], d_1)$ , let  $U$  be the set

$$U = \{f : f(1) = 1\}.$$

Determine whether or not  $U$  is closed, giving reasons for your answer.

(5 marks)

- 3 (i) Let  $(X, d)$  be a metric space.
- (a) Define what it means for a sequence  $(x_n)$  to be a *Cauchy sequence*.
  - (b) Define what it means for  $(X, d)$  to be *complete*.
  - (c) Show that a convergent sequence is Cauchy. (7 marks)

- (ii) Consider  $C[0, 1]$  with the metric  $d_\infty$  defined by

$$d_\infty(f, g) = \sup\{|f(x) - g(x)| : x \in [0, 1]\}.$$

Define  $(f_n)$ , for  $n \in \mathbb{N}$ , by

$$f_n(x) = 1 + \frac{x}{2} + \frac{x^2}{2^2} + \cdots + \frac{x^n}{2^n},$$

for  $x \in [0, 1]$ .

- (a) For  $m > n$ , show that  $f_m(x) - f_n(x) \geq 0$  for all  $x \in [0, 1]$ .  
 Show also that  $f_m - f_n$  is strictly increasing on  $(0, 1]$ .  
 Hence, or otherwise, show that, for  $m > n$ ,

$$d_\infty(f_n, f_m) = f_m(1) - f_n(1) = \frac{1}{2^n} - \frac{1}{2^m}.$$

You may use, if you wish, the formula

$$1 + r + r^2 + \cdots + r^k = \frac{1 - r^{k+1}}{1 - r}, \quad r \neq 1.$$

- (b) Show that  $(f_n)$  is a Cauchy sequence in  $(C[0, 1], d_\infty)$ . (18 marks)

- 4** (i) Let  $X = [0, 2\pi) \subseteq \mathbb{R}$  have the subspace metric from the usual metric on  $\mathbb{R}$  and let  $Y = \{(x^1, x^2) \in \mathbb{R}^2 : (x^1)^2 + (x^2)^2 = 1\}$  have the subspace metric from the usual Euclidean metric  $d_2$ .

You may assume that  $f: X \rightarrow Y$ ,  $f(t) = (\cos(t), \sin(t))$  is continuous and a bijection.

Show that the inverse function  $f^{-1}: Y \rightarrow X$  is not continuous.

(4 marks)

- (ii) (a) Let  $(X, d_X)$  be a metric space. Define what it means for a subset  $A \subseteq X$  to be *compact*.
- (b) Prove that a compact subset of a metric space is closed.
- (c) Prove that a closed subset of a compact metric space is compact.

(9 marks)

- (iii) Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces, and let  $f: X \rightarrow Y$  be a continuous function.

- (a) Prove that if  $A \subseteq X$  is compact, then  $f(A) \subseteq Y$  is compact.
- (b) Now assume that  $X$  is compact and that  $f: X \rightarrow Y$  is a continuous bijection.

Prove that the inverse function  $f^{-1}: Y \rightarrow X$  is continuous.

(12 marks)

[You should state clearly, but need not prove, any result(s) from the course that you use.]

**End of Question Paper**