The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2019–20**

Complex Analysis **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Let $z, w \in \mathbb{C}$. State, without proof, the triangle inequalities for $|z+w|$. ***(2 marks)***

(ii) Show that for all $z \in \mathbb{C}$ such that $|z-2i| < 1$,

$$\frac{\sinh(1)}{\sqrt{5}+1} \leq \left|\frac{\sin(z)}{z+1}\right| \leq \frac{\cosh(3)}{\sqrt{5}-1}$$

(7 marks)

(iii) State, without proof, the Cauchy-Riemann equations. Your statement should include conditions under which they hold. ***(3 marks)***

(iv) Let $g(z) = \mathrm{Im}(z)^2$. Find all points where $g(z)$ is differentiable. ***(5 marks)***

(v) For each of the $u(x,y)$ given below, decide whether there is a function $f(z)$ analytic on $\mathbb{C}$ such that $\mathrm{Re}[f(x+iy)] = u(x,y)$. When f exists find an explicit expression for it in terms of z.

(a) $u(x,y) = x^3y - xy^3$

(b) $u(x,y) = \sinh(xy)$

(8 marks)

2 For parts (i) and (ii), let $\beta(t) = e^{it}, \quad -\pi/2 \le t \le \pi/2$.

(i) Find $\displaystyle\int_\beta \mathrm{Im}(z)^2 dz$. *(6 marks)*

(ii) Find $\displaystyle\int_\beta \frac{\cos(z)}{\sin^2(z)} dz$. *(4 marks)*

(iii) State, without proof, Cauchy's Theorem and Cauchy's Integral Formulae for a function and for its derivatives. Your statement should include conditions under which the results are valid. *(6 marks)*

(iv) Let γ be the square contour with vertices $0,\ 2-2i,\ 4,\ 2+2i$, described in the anti-clockwise direction. Without using the Residue Theorem, evaluate

(a) $\displaystyle\int_\gamma \frac{\cos \pi z}{2z+\pi}\, dz\,,$

(b) $\displaystyle\int_\gamma \frac{z^3 \sin^2(z)}{z^2-4}\, dz\,,$

(c) $\displaystyle\int_\gamma \frac{e^z + z^3}{(2z-4-i\pi)^5}\, dz\,,$

(9 marks)

3 (i) (a) State, without proof, Liouville's Theorem. *(2 marks)*

(b) The functions f and g are analytic in $\mathbb{C}$ and satisfy the relation $|f(z)+g(z)| < |f(z)-g(z)|$ for all $z \in \mathbb{C}$. Show that there is a constant a such that $g(z) = af(z)$ for all $z \in \mathbb{C}$. *(6 marks)*

(ii) Expand $h(z) = \dfrac{1}{1-z}$ in a Taylor series around $z = i$. *(4 marks)*

(iii) Using the ratio test, find the radius of convergence of your series for h in the previous part. Explain how you could have gotten this answer by considering the singularities of h. *(4 marks)*

(iv) Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(3n)!}{n!n!} \frac{z^{2n}}{3^n}$$

(4 marks)

(v) Prove that if a function g has a zero of order k at α, then $\dfrac{1}{g}$ has a pole of order k at α. *(5 marks)*

4 (i) Consider the three functions h, j, k defined as follows:

$$h(z) = \frac{1-e^z}{1+e^z}, \quad j(z) = \frac{ze^z}{(z-1)^4}, \quad k(z) = \left[2z\cosh\left(\frac{1}{z+1}\right)\right]^2$$

For each of the three functions h, j, k:

(a) Find all of its singularities in $\mathbb{C}$

(b) Classify its singularities, giving reasons for your answers

(c) Find the residue at each of the singularities.

(15 marks)

(ii) Prove that

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+x+1} dx = \frac{2\pi}{\sqrt{3}} e^{-\sqrt{3}/2} \cos(1/2)$$ ***(10 marks)***

End of Question Paper