The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2019-20**

Fields **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) State the Subfield Criterion. ***(3 marks)***

(ii) Prove that the intersection $K_1 \cap K_2$ of two subfields K_1 and K_2 of a field L is also a subfield of L. ***(2 marks)***

(iii) For each of the subsets J_1 and J_2 of $\mathbb{C}$ specified below determine, with justification, whether it is a subfield of $\mathbb{C}$:

(a) $J_1 = \{a + b\sqrt{-2} : a, b \in \mathbb{Q}\}$. ***(5 marks)***

(b) $J_2 = \{a + b\sqrt{3} + c\sqrt{5} : a, b, c \in \mathbb{Q}\}$. ***(5 marks)***

(iv) Show that the following subfields L_1 and L_2 of $\mathbb{R}$ are equal where

$$L_1 = \mathbb{Q}(\sqrt{2}, \sqrt{3}) \text{ and } L_2 = \mathbb{Q}\left(\sqrt{2}+\sqrt{3}, \sqrt{2}-\sqrt{3}, \frac{5\sqrt{2}+7\sqrt{3}}{2\sqrt{2}+3\sqrt{3}}\right).$$

(5 marks)

(v) Is the element $\alpha = 2\sqrt{2} - 3\sqrt{3} + 1$ equal to 0? Justify your response. ***(5 marks)***

2 (i) State Gauss' Lemma. *(2 marks)*

(ii) Give a definition of the content $c(f)$ of a polynomial $f \in \mathbb{Z}[x]$ and prove Gauss' Lemma. *(6 marks)*

(iii) Let $K \subseteq L$ be a field extension and $a \in L$ be an algebraic element over the field K.

(a) Define the *minimal polynomial* $m(x) \in K[x]$ of the element a over K and prove that the polynomial $m(x)$ irreducible over K. *(6 marks)*

(b) Suppose that $n = \deg(m(x))$. Prove that the set of elements $1, a, a^2, \ldots, a^{n-1}$ form a basis for the vector space $K(a)$ over the field K. *(8 marks)*

(c) Find the minimal polynomial $m(x) \in \mathbb{Q}(\sqrt{3})[x]$ of the element $a = \sqrt{3} + 3\sqrt{2}$ over the field $\mathbb{Q}(\sqrt{3})$. *(3 marks)*

3 (i) (a) State Eisenstein's Irreducibility Criterion. *(2 marks)*

(b) Prove Eisenstein's Irreducibility Criterion. *(8 marks)*

(c) Show that the polynomial $13x^{10} - 6x^7 - 36x^3 + 12$ is irreducible in $\mathbb{Q}[x]$. *(2 marks)*

(ii) (a) State the 'Shifted Eisenstein's Irreducibility Criterion'. *(3 marks)*

(b) Prove the 'Shifted Eisenstein's Irreducibility Criterion'. *(7 marks)*

(c) Show that the polynomial $f(x) = x^2 + 7x - 5$ is irreducible in $\mathbb{Q}[x]$ by using the 'Shifted Eisenstein's Irreducibility Criterion' (or otherwise). *(3 marks)*

4 (i) Give a definition of a constructible point $P \in \mathbb{R}^2$. *(2 marks)*

(ii) State Standard Constructions I-IV (for each Standard Construction there is no need to give an algorithm for constructing points/lines). *(4 marks)*

(iii) Let $(a, b) \in \mathbb{R}^2$. Give a criterion for constructibility of the point (a, b) (via quadratic fields). *(3 marks)*

(iv) Prove the criterion. *(8 marks)*

(v) Using **only** the criterion show that the point $(\sqrt{1 + \sqrt[4]{5}}, 0)$ is a constructible point. *(4 marks)*

(vi) Is the point $(\frac{3}{13}, \sqrt[3]{5})$ constructible? Justify your response. *(4 marks)*

End of Question Paper