Data provided: list of formulae

MAS336

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — **Autumn Semester 2019–20**

Differential Geometry — **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

A list of formulae is provided on the last two pages.

Please leave this exam paper on your desk
Do not remove it from the hall

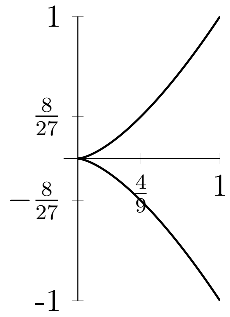
Registration number from U-Card (9 digits)
to be completed by student

Blank

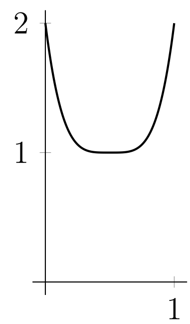
1 State whether each of the following assertions is true or false and give *careful* justification of your answer. Results used from the course should be precisely stated.

(i) The line $\{(x, y) \mid x = 0\} \subset \mathbb{R}^2$ is the image of a parametrized curve. ***(5 marks)***

(ii) The following picture is the image of a *smooth* parametrized curve. ***(5 marks)***

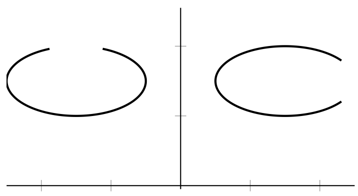


(iii) The following picture is the image of a *unit speed* curve $\gamma_{\mathrm{a}} \colon \left]0, 1\right[\to \mathbb{R}^2$. [You might wish to consider the length of the curve.] ***(5 marks)***



(iv) There is a parametrized curve $\gamma_{\mathrm{b}} \colon \mathbb{R} \to \mathbb{R}^2$ such that the curvature is given by $\kappa(t) = -\cos(t)$. ***(5 marks)***

(v) The two pictured curves are the images of two unit-speed parametrized curves $\left]\alpha, \beta\right[\to \mathbb{R}^2$ (for some $\alpha < \beta$) with the same curvature function. ***(5 marks)***



2 Suppose that $\gamma\colon\ ]\alpha,\beta[\ \to \mathbb{R}^2$ is a regular curve. The derivative $\gamma'\colon\ ]\alpha,\beta[\ \to \mathbb{R}^2$ may be expressed in polar coordinates as

$$\gamma'(t) = \Big(r(t)\cos\big(\theta(t)\big), r(t)\sin\big(\theta(t)\big)\Big)$$

for functions $r\colon\ ]\alpha,\beta[\ \to \mathbb{R}_{\geq 0}$ and $\theta\colon\ ]\alpha,\beta[\ \to \mathbb{R}$.

(i) If γ represents the trajectory of a particle in the plane, then what, in words, does r represent? ***(2 marks)***

(ii) Show that the curvature is given by

$$\kappa(t) = \frac{\theta'(t)}{r(t)}. \qquad (*)$$

What property of the curve allows you to deduce that this expression is well-defined? ***(9 marks)***

(iii) Use the above expression $(*)$ to calculate, for $R > 0$, the curvature of the parametrized curve

$$\gamma_2\colon \mathbb{R} \to \mathbb{R}^2; \quad \gamma_2(t) := \big(R\cos(2t), R\sin(2t)\big).$$

[Hint: Remember that $\cos(\theta) = \sin(\theta + \pi/2)$ for all $\theta \in \mathbb{R}$.]

Comment on your answer. ***(8 marks)***

(iv) Returning to the general case of $\gamma\colon\ ]\alpha,\beta[\ \to \mathbb{R}^2$ as above, suppose that there are $t_0, t_1 \in\ ]\alpha,\beta[$ such that $\gamma'(t_1) = \lambda\gamma'(t_0)$ for some $\lambda > 0$. Show that there is an $n \in \mathbb{Z}$ such that

$$\int_{t_0}^{t_1} r(t)\kappa(t)\mathrm{d}t = 2\pi n.$$

Give a geometric interpretation of the integer n. ***(6 marks)***

3 (i) Sketch both the image of the parametrized curve

$$\gamma\colon\ ]0,\infty[\ \to \mathbb{R}^3; \quad t \mapsto (\sqrt{t}, 0, \sqrt{t})$$

and the resulting surface of revolution obtained by rotating the curve around the z-axis. Indicate any notable feature. ***(5 marks)***

(ii) Give the standard parametrization σ of the parametrized surface of revolution of the parametrized curve γ. Explain why this is a regular parametrized surface: you may use any facts from the lectures provided they are clearly stated. ***(5 marks)***

(iii) For the parametrized surface σ, calculate the first fundamental form and its determinant. ***(5 marks)***

(iv) For $i \in \{1, 2, 3, \dots\}$, consider the small disk D_i of radius 0.1 centred at $(2i, 0) \in\ ]0,\infty[\ \times \mathbb{R}$. How does the area of the image $\sigma(D_i)$ relate to the area of D_i? ***(5 marks)***

4 Consider the hyperboloid, given by the parametrized surface

$$\rho\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}^3; \quad (t,\theta)\mapsto(\cosh(t)\cos(\theta),\cosh(t)\sin(\theta),\sinh(t)).$$

The matrix representing the first fundamental form of ρ at (t,θ) is given by

$$\mathrm{I}=\begin{pmatrix}\cosh(2t) & 0\\ 0 & \cosh^2(t)\end{pmatrix}.$$

(i) Show that the preferred unit normal and the second fundamental form are given respectively by the following expressions:

$$\mathbf{n}=\tfrac{1}{\sqrt{\cosh(2t)}}(-\cosh(t)\cos(\theta),-\cosh(t)\sin(\theta),\sinh(t));$$
$$\mathrm{II}=\frac{1}{\sqrt{\cosh(2t)}}\begin{pmatrix}-1 & 0\\ 0 & \cosh^2(t)\end{pmatrix}.$$

(13 marks)

(ii) Find the principal curvatures and corresponding (normalized) principal vectors. ***(10 marks)***

(iii) Show that ρ parametrizes the level set $S_1 := \{(x,y,z) \mid x^2+y^2-z^2=1\}$. [You may assume that $\theta\mapsto(\cos\theta,\sin\theta)$ for $t\in\mathbb{R}$ parametrizes the unit circle.] ***(5 marks)***

(iv) The level set $S_2 := \{(x,y,z) \mid x^2+y^2-z^2=-1\}$ is *not* the image of a surface of revolution of a parametrized curve as defined in this course. Explain why not. ***(2 marks)***

End of Question Paper

List of formulae

- The inverse of a 2×2-matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with coefficients in $\mathbb{R}$ and $ad - bc \neq 0$ is
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$
- The cross-product of two vectors $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$ is
$$v_1 \times v_2 = (y_1 z_2 - z_1 y_2, z_1 x_2 - z_2 x_1, x_1 y_2 - x_2 y_1) \in \mathbb{R}^3.$$
- The angle θ between two vectors v_1 and $v_2 \in \mathbb{R}^3$ is given by
$$\cos\theta = \frac{v_1 \cdot v_2}{\|v_1\| \, \|v_2\|}.$$

Here are some hyperbolic identities.

- $\cosh^2(x) + \sinh^2(x) = \cosh(2x)$
- $\cosh^2(x) - \sinh^2(x) = 1$

Inverse hyperbolic functions are given by the following.

- $\sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$
- $\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1})$ for $x \geq 1$
- $\tanh^{-1}(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$ for $|x| < 1$

Derivatives of some functions are given by the following.

- $\frac{d}{dx} \operatorname{sech}(x) = -\tanh(x) \operatorname{sech}(x)$
- $\frac{d}{dx} \tanh(x) = \operatorname{sech}^2(x)$
- $\frac{d}{dx} \sinh^{-1}(x) = \frac{1}{\sqrt{x^2+1}}$
- $\frac{d}{dx} \cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}}$ for $x > 1$
- $\frac{d}{dx} \tanh^{-1}(x) = \frac{1}{1-x^2}$ for $|x| < 1$
- $\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2}$

Some relations between trigonometric functions are the following.

- $\sin(\tan^{-1} x) = \frac{x}{\sqrt{1+x^2}}$
- $\cos(\tan^{-1} x) = \frac{1}{\sqrt{1+x^2}}$
- $\cos(\theta) = \sin(\theta + \pi/2)$

For a parametrized curve $\gamma\colon\ ]\alpha,\beta[\ \to \mathbb{R}^2$, $\gamma(t) = (x(t), y(t))$ we have the following.

- The arc length from $\gamma(a)$ to $\gamma(b)$, $\alpha < a \le b < \beta$ is
$$\int_a^b \|\dot\gamma(t)\|\, dt.$$
- The curvature of γ at t is
$$\kappa(t) = \frac{\ddot\gamma(t) \cdot J(\dot\gamma(t))}{\|\dot\gamma(t)\|^3} = \frac{x'(t)y''(t) - y'(t)x''(t)}{[x'(t)^2 + y'(t)^2]^{3/2}},$$
where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the anti-clockwise rotation of angle $\pi/2$.

For a parametrized surface $\sigma\colon U \to \mathbb{R}^3$, with U an open set in $\mathbb{R}^2$ the following hold.

- The matrix of the first fundamental form is given by
$$\mathrm{I}_{(u,v)} = \begin{pmatrix} E(u,v) & F(u,v) \\ F(u,v) & G(u,v) \end{pmatrix}$$
for all $(u,v) \in \mathbb{R}^2$, with $E = \sigma_u \cdot \sigma_u$, $F = \sigma_u \cdot \sigma_v$ and $G = \sigma_v \cdot \sigma_v$.
- The area of the domain $\sigma([\alpha_1, \beta_1] \times [\alpha_2, \beta_2])$, for $[\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \subseteq U$ is given by
$$\int_{u=\alpha_1}^{\beta_1} \int_{v=\alpha_2}^{\beta_2} \sqrt{EG - F^2}\, dv\, du$$
- The preferred unit normal vector along σ is given by $\mathbf{n}\colon U \to \mathbb{R}^3$,
$$\mathbf{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}.$$
- The matrix of the second fundamental form of σ at $(u,v) \in U$ is
$$\mathrm{II}_{(u,v)} = \begin{pmatrix} L(u,v) & M(u,v) \\ M(u,v) & N(u,v) \end{pmatrix}$$
where $L = \sigma_{uu} \cdot \mathbf{n}$, $M = \sigma_{uv} \cdot \mathbf{n}$ and $N = \sigma_{vv} \cdot \mathbf{n}$.
- The Weingarten matrix of σ is
$$W = \mathrm{I}^{-1}\, \mathrm{II}\,.$$
- The Gaussian curvature is
$$K = \det W.$$

The Brioschi formula:
$$K = \frac{\begin{vmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix}}{(EG - F^2)^2}.$$