The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2019–20**

Game Theory **2 hours and 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 (i) Alice and Bob play a game given in normal form as follows

	I	II	III
A	4, 1	0, 2	1, 0
B	4, 0	3, 3	3, 1
C	1, 1	0, 0	3, 2

(a) Find all pure-strategy Nash equilibria of this game. ***(4 marks)***

(b) Is there a mixed-strategy Nash equilibrium where Alice plays A and B with positive probability but does not play C? Justify your answer. ***(2 marks)***

(c) What are Bob's mixed-strategy best responses against Alice's mixed strategy $(0, 1/2, 1/2)$? ***(2 marks)***

(d) Find all mixed-strategy Nash equilibria where Alice plays B and C with positive probability but does not play A. Justify your answer. ***(5 marks)***

(ii) Alice and Bob form a duopoly in space tourism, and both need to decide on the sums of their investments in space exploration for the following year. If Alice invests a million pounds and Bob invests b million pounds, their profits will be $a(3 - a - b)$ and $b(5 - a - b)$ million pounds, respectively.

(a) If both investment decisions are made independently and simultaneously, which decisions will result in a Nash equilibrium? ***(7 marks)***

(b) If Bob needs to decide the amount of his investment after he learns about Alice decision, which sums will be invested? ***(5 marks)***

2 (i) A group of $n \geq 2$ friends are unable to communicate. Each one of them faces the decision whether to attend their favourite pub that evening. Going to the pub and not finding any of the other $n-1$ friends there incurs a cost of 10 units of utility. Meeting a friend at the pub provides 5 units of utility. Not going to the pub results in no change in utility.

(a) Describe this set-up as a game in normal form and describe all its pure-strategy Nash equilibria. ***(8 marks)***

(b) Consider a mixed strategy $\mathcal{S}$ where a friend goes to the pub with some fixed probability $0 < p < 1$. Use the indifference principle to find the value of p for which the strategy profile where each friend following $\mathcal{S}$ is a Nash equilibrium. ***(5 marks)***

(ii) Alice and Bob face the following game

	l	r
U	4, 8	2, 0
D	-1, 5	7, 3

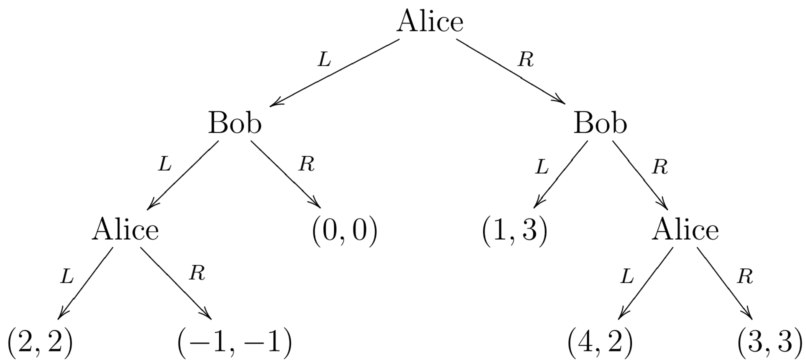
and choose to negotiate an outcome, with the knowledge that, if they fail to strike a deal, utilities of 5 will be imposed on both.

(a) Sketch the cooperative payoff region of the game. ***(3 marks)***

(b) Describe parametrically the payoffs in the cooperative payoff region that satisfy both the *Individual Rationality* and *Pareto Optimality* conditions. ***(4 marks)***

(c) Find the Nash Bargain of this set-up. ***(5 marks)***

3 (i) Alice and Bob play the sequential game described by the following tree



where utilities of a for Alice and b for Bob are denoted (a, b).

(a) Solve this game using backward induction. ***(3 marks)***

(b) Describe the game in strategic form, find all its pure-strategy Nash equilibria and indicate which of these is subgame perfect. ***(10 marks)***

(ii) Consider the following 2-player sequential game $G(N_1, N_2)$. The game starts with two piles of tokens of sizes $N_1, N_2 \geq 1$ and the players alternate in making moves. Each move consists of choosing a pile and removing any positive number of tokens from that pile. The player who takes the last remaining tokens wins.

Prove that the second player has a strategy that guarantees him victory if and only if $N_1 = N_2$. ***(12 marks)***

4 (i) Consider a 2-player game given in strategic form as (S, T, u_1, u_2).

(a) Define the *minimax values* of both players. ***(2 marks)***

(b) Define the *cooperative payoff region* of the game. ***(2 marks)***

(ii) Consider the 2-person game G given in tabular form as follows

	A	B	C
I	-3, -3	5, 0	0, 0
II	-3, 2	1, -1	0, 5

(a) Sketch the cooperative payoff region of G and show that it contains the point $(2, 2)$. ***(4 marks)***

(b) Consider now the game G^∞ which consists of playing G repeatedly, and where the payoffs of the infinite game are the average payoffs. Describe, without proof, a Nash equilibrium that results in an average payoff of 2 for both players. ***(4 marks)***

(iii) Alice and Bob play one of the two following games

Game I

	L	R
U	2, 2	0, 0
D	0, 0	4, 4

Game II

	L	R
U	0, 2	2, 0
D	4, 0	0, 4

(a) Suppose that Alice knows which game is being played, but Bob does not, and he believes that each game is played with probability 1/2. Model this as a Bayesian game and find a Bayes-Nash equilibrium of this game. ***(9 marks)***

(b) Suppose now that neither Alice nor Bob know which game is being played, and both believe that each game is played with probability 1/2. Find a Bayes-Nash equilibrium of this game. ***(4 marks)***

End of Question Paper