The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2019–20**

Financial Mathematics **2 hours and 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
Do not remove it from the hall

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1 (i) Consider the following five bonds with face value of £100:

Time to maturity (in years)	Annual interest (paid every 6 months)	Bond price (in £)
0.25	0	99.0
0.5	0	97.8
1.0	0	95.5
1.5	8%	104.5
2.0	12%	113.0

(a) Find the 0.25, 0.5 and 1-year spot interest rates. ***(2 marks)***

(b) Use the bootstrap method to find the 1.5 and 2-year spot interest rates. ***(5 marks)***

(ii) Consider a forward contract on foreign currency. By considering two portfolios involving positions on the forward contract and domestic and foreign currency deposits show that

$$Fe^{-rT} = Se^{-r_f T},$$

where F is the forward exchange rate (i.e., domestic currency paid for 1 unit of foreign currency paid on delivery), S is the spot price of the asset (i.e., domestic currency paid at the present for 1 unit of foreign currency), T is the time to maturity of the forward contract (in years), r is the T-year spot interest rate for domestic currency and r_f is the T-year spot interest rate for the foreign currency. ***(8 marks)***

(iii) Suppose that the domestic one-year and three-year spot interest rates are 4% and 5% respectively.

(a) What is the forward rate for the period starting in one year and ending in three years? ***(1 mark)***

(b) Suppose that you are offered by a risk free institution the opportunity to deposit or borrow £10,000,000 in one year for a period of two years earning an interest rate of 5%. Describe in detail an arbitrage opportunity available to you. ***(9 marks)***

2 (i) Consider two European call options with the same underlying asset and same expiration date in T-years, and with strike prices $X_1 < X_2$. Consider also a portfolio which is long one call option with strike price X_1 and short one call option with strike X_2. Let r be the T-year spot interest rate.

Sketch the payoff function of the portfolio. ***(5 marks)***

(ii) Let c be the price of a European call option on stock which pays no dividends; let X be the strike price of the option, let S be the spot price of the stock, let T be the time to the option expiration (in years) and let r be the T-year spot interest rate. By comparing a portfolio consisting of options and cash, to a portfolio consisting of one share, prove that

$$c \geq S - Xe^{-rT}.$$ ***(4 marks)***

(iii) Let c and p be the prices of a European call option and put option, respectively, on stock which pays no dividends; let X be the strike price of both options, let S be the spot price of the stock, let T be the time to the options expiration (in years) and let r be the T-year spot interest rate. By comparing two portfolios consisting of options and cash, prove that

$$c + Xe^{-rT} = p + S.$$ ***(10 marks)***

(iv) Suppose that c_1, c_2, and c_3 are the prices of European call options with strike prices X_1, X_2, and X_3 respectively, where $X_3 > X_2 > X_1$ and $X_3 - X_2 = X_2 - X_1$. We are told that all options have the same expiration date. Show that

$$c_2 \leq \frac{1}{2}(c_1 + c_3).$$ ***(6 marks)***

3 (i) Explain the principle of risk-neutral valuation. ***(3 marks)***

(ii) The price of a stock which pays no dividends is currently £20. Over each of the next three 1-year periods the stock price will either increase by 10% or decrease by 10%. Suppose that all interest rates are constant and equal to 3%.

(a) Use a binomial tree to find the price of a three-year American put option on this stock with strike price £20. ***(11 marks)***

(b) Describe all circumstances when a rational investor should exercise the option. ***(3 marks)***

(iii) Consider a European call option on an asset with spot price S, expiration time in one year, and strike price S. You are told that a year from now the price of the asset will have either risen by a factor of u ($u > 1$) or will have decreased by a factor of $\frac{1}{u}$. If the interest rate on one-year deposits is $r > 0$, show that the price of the European call option is an increasing function of the variable u. ***(8 marks)***

4 (i) State Ito's Lemma. ***(5 marks)***

(ii) In this question we consider a European call option and a European put option, both with the same underlying stock, the same strike price X and the same maturity time $T > 0$. For any time $0 \leq t \leq T$, let $S_t = S$ denote the spot price of the underlying stock, let $c(S, t)$ be the price of European call and let $p(S, t)$ be the price of European put option at time t. Assume that all spot interest rates are constant and equal to r. Assume also that the underlying stock price follows the Ito process

$$dS = rSdt + \sigma SdB$$

(a) Verify that $f(S, t) = Xe^{-r(T-t)} - S$ is a solution of the Black-Scholes partial differential equation

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf.$$

(2 marks)

(b) Deduce that $g(S, t) = Xe^{-r(T-t)} - S + c(S, t) - p(S, t)$ is also a solution of the Black-Scholes partial differential equation. ***(3 marks)***

(c) By considering the case $t = T$, show that (b) implies that

$$c(S, t) + Xe^{-r(T-t)} = p(S, t) + S.$$ ***(6 marks)***

(iii) For any time $0 \leq t \leq T$, let $S_t = S$ denote the spot price of a stock. Assume that all spot interest rates are constant and equal to r. We also assume that the stock price S_t follows geometric Brownian motion $dS = rSdt + \sigma SdB$.

(a) Show that the process $S_t^n, n \geq 2$ also follows geometric Brownian motion, and find its drift. ***(3 marks)***

(b) Now consider a derivative, on the same stock, that pays off S_T^n at time T, where S_T is the stock price at that time. If you are told that the price of the derivative at time t ($t \leq T$) has the form

$$h(t)S_t^n,$$

where S_t is the stock price at time t, h is a function only of t, and that the price of the derivative satisfies the Black-Scholes equation,

(α) find the differential equation that $h(t)$ satisfies. ***(2 marks)***

(β) find the value of $h(T)$, which plays the role of a "boundary" condition, for the above differential equation ***(2 marks)***

(γ) show that

$$h(t) = e^{[\frac{1}{2}\sigma^2 n(n-1) + r(n-1)](T-t)},$$

where r is the risk-free interest rate and σ is the stock price volatility. ***(2 marks)***

End of Question Paper