Data provided: Formula sheet

MAS380

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2019–20**

Computational Engineering Mathematics **Three hours**

Marks will be awarded for your best FOUR answers.
The maximum possible mark for the paper is 100.

1 (a) Second order linear partial differential equations for $u(x,y)$ can be written in the general form

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G,$$

where A, B, C, D, E, F and G are functions of x and y.

(i) What are the conditions on A, B, C, D, E, F and G which determine whether the equation is elliptic, parabolic or hyperbolic?

(2 marks)

(ii) Identify the regions of the (x,y) plane where the equation

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial y}+x\frac{\partial^2 u}{\partial y^2}=xy$$

is elliptic, the regions where it is parabolic, and the regions where it is hyperbolic. ***(5 marks)***

 Question 1 continued on next page

1 (continued)

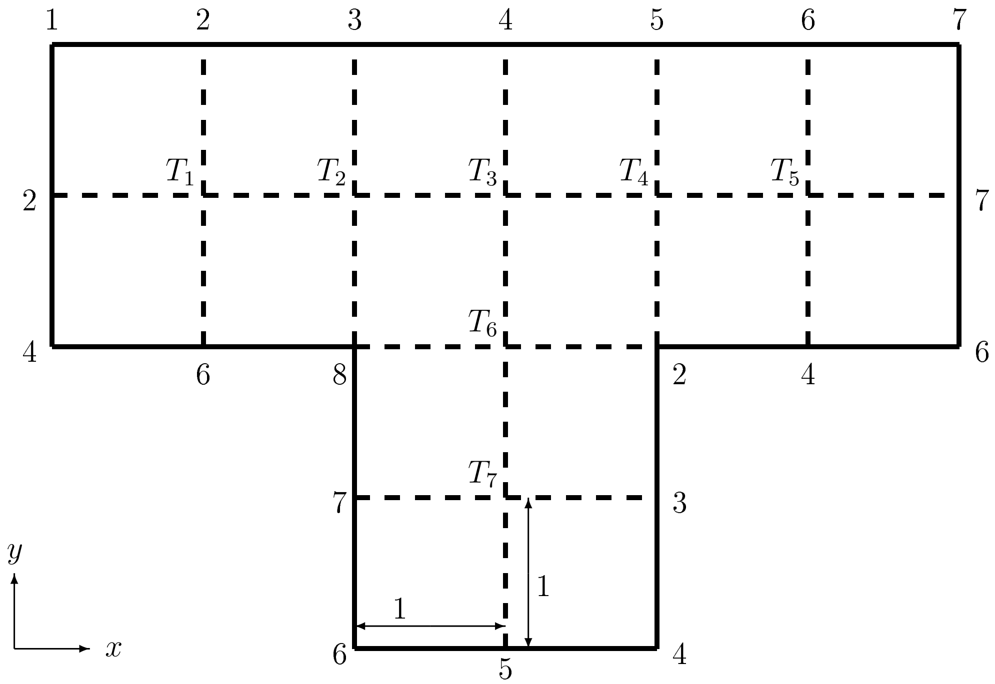
(b) $T(x, y)$ satisfies the boundary conditions indicated in the figure, and the differential equation

$$\frac{\partial^2 T}{\partial x^2} + y\frac{\partial^2 T}{\partial y^2} = 0.$$

The figure (where the top left corner is $x = 0$, $y = 4$) shows the solution domain, divided into intervals of length $\Delta x = 1$ in the x direction, and length $\Delta y = 1$ in the y direction.

(i) In the interior of the region is this differential equation elliptic, parabolic or hyperbolic? ***(2 marks)***

(ii) Use the finite difference formulae on the formula sheet to derive the finite difference equations required to find estimates of the nodal values $T_1, T_2, \ldots, T_7$, expressing these in the form $A\mathbf{T} = \mathbf{b}$, where A is a 7×7 matrix, $\mathbf{T} = [T_1, T_2, T_3, T_4, T_5, T_6, T_7]^T$ and $\mathbf{b}$ is a constant vector. (You do NOT need to try to solve these equations.) ***(16 marks)***



2 The temperature $T(x,t)$ satisfies the heat conduction equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \qquad (0 \leqslant x \leqslant 1,\ t > 0),$$

and the boundary and initial conditions

$$T(0,t) = 10, \qquad \frac{\partial}{\partial x} T(x,t)\bigg|_{x=1} = 2, \qquad T(x,0) = 10 + x^2,$$

where all units are in terms of °C, m and s.

(a) If $T_{i,j} = T(x_i, t_j)$, with $i = 0$ and $i = 5$ corresponding to $x = 0$ and $x = 1$, respectively, and $j = 0$ corresponding to $t = 0$, use backward differences for time derivatives and central differences for space derivatives to derive the equations for $T_{1,1}$, $T_{2,1}$, $T_{3,1}$, $T_{4,1}$ and $T_{5,1}$. Take $\Delta t = 0.04$. (You do NOT need to solve the equations.) ***(15 marks)***

(b) Show from your answer to part (a) that the Jacobi equations to find the $(k+1)$th iteration from the kth iteration are

$$\begin{bmatrix} T_{1,1} \\ T_{2,1} \\ T_{3,1} \\ T_{4,1} \\ T_{5,1} \end{bmatrix}^{(k+1)} = \frac{1}{3}\begin{bmatrix} 20.04 \\ 10.16 \\ 10.36 \\ 10.64 \\ 11.8 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} T_{1,1} \\ T_{2,1} \\ T_{3,1} \\ T_{4,1} \\ T_{5,1} \end{bmatrix}^{(k)}$$

(10 marks)

3 (a) A rectangular surface on a positive y-plane is defined by $0 \leqslant x \leqslant 2d$, $y = 0$ and $0 \leqslant z \leqslant d$ (where d is a constant length, and all lengths are measured in m). The stress tensor depends on x and is given by (in units of Pa)

$$[\sigma] = \frac{1}{d^2}\begin{bmatrix} 2\hat{z} & 3\hat{z}^2 & 8\hat{z}^3 \\ 3\hat{z}^2 & 6\hat{z}^2 & 4\hat{z} \\ 8\hat{z}^3 & 4\hat{z} & 2\hat{z} \end{bmatrix},$$

where $\hat{z} = z/d$.

Show that the total stress force $\mathbf{f}$ on the rectangular surface is

$$\mathbf{f} = 2\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \text{N}.$$

(7 marks)

(b) The matrix relating the *engineering* strains to the stresses for an isotropic material is given by

$$\begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

where $\lambda = \dfrac{K(9K - 3E)}{9K - E}$ and $\mu = \dfrac{3KE}{9K - E}$. If $E = 32.3\,\text{GPa}$, $K = 27.1\,\text{GPa}$, $\varepsilon_{xx} = 634 \times 10^{-6}$, $\varepsilon_{yy} = -0.42\varepsilon_{xx}$, $\varepsilon_{zz} = 0.31\varepsilon_{xx}$, $\varepsilon_{xy} = -173 \times 10^{-6}$, $\varepsilon_{yz} = 321 \times 10^{-6}$ and $\varepsilon_{zx} = 97 \times 10^{-6}$ at a particular point in the material, calculate the stress at that point, giving your answers to 3 significant figures. ***(11 marks)***

(c) Let the i-component of the body force per unit volume in a solid body be F_i, and the components of the stress tensor be σ_{ji}. Consider a volume V in the solid body which is enclosed by a surface S.

If the body is in equilibrium, derive an equation (involving integrals) for the total force on the volume V.

Using Gauss' theorem in the form

$$\int_V \frac{\partial \sigma_{ji}}{\partial x_j}\, dV = \int_S \sigma_{ji}\hat{n}_j\, dS,$$

where dS is a surface area element and $\hat{\mathbf{n}}$ is a unit vector normal to S and pointing out of V, deduce that

$$\frac{\partial \sigma_{ji}}{\partial x_j} + F_i = 0$$

at every point in the body. ***(7 marks)***

4 An incompressible fluid of constant density ρ flows steadily between slightly porous horizontal solid boundaries at $z = 0$ and $z = a$, where a is a positive constant. Fluid percolates in through one wall and out through the other at the same constant speed u, while the main flow is along the channel at speed $U(z)$. The total fluid velocity can be taken to be

$$\mathbf{v} = U(z)\mathbf{i} + u\mathbf{k},$$

(where $\mathbf{i}$ and $\mathbf{k}$ are the unit vectors in the x and z directions, respectively) with boundary conditions

$$U = 0 \qquad \text{at } z = 0 \text{ and at } z = a.$$

Body forces can be ignored, and the fluid is Newtonian, so

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{v},$$

where μ is the viscosity.

(a) Deduce that

$$\frac{\partial p}{\partial x} = -G$$

for some constant G. ***(6 marks)***

(b) Derive the solution

$$U(z) = A + Be^{uz/\nu} + \frac{Gz}{\rho u},$$

where A and B are constants, and $\nu = \mu/\rho$ is the kinematic viscosity. ***(8 marks)***

(c) Use the boundary conditions to find the values of A and B, and hence show that

$$U(z) = \frac{Ga}{\rho u} \left\{ \frac{z}{a} - \frac{1 - e^{uz/\nu}}{1 - e^{ua/\nu}} \right\}.$$

(6 marks)

(d) If u is non-zero and $G > 0$, deduce that $U > 0$ midway between the solid boundaries. ***(5 marks)***

5 (a) In index notation the i-component of the vector product of $\mathbf{u}$ and $\mathbf{v}$ may be expressed as

$$(\mathbf{u} \times \mathbf{v})_i = \varepsilon_{ijk} u_j v_k,$$

and the i-component of the curl of a vector field $\mathbf{F}$ may be expressed as

$$(\nabla \times \mathbf{F})_i = \varepsilon_{ijk} \frac{\partial F_k}{\partial x_j},$$

where ε_{ijk} is the Levi-Civita tensor.

You may assume that

$$\varepsilon_{kij}\, \varepsilon_{klm} = \delta_{il}\, \delta_{jm} - \delta_{im}\, \delta_{jl}.$$

Show, using index notation, that

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = \mathbf{u}(\nabla \cdot \mathbf{v}) + (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} - \mathbf{v}(\nabla \cdot \mathbf{u}).$$

(11 marks)

(b) The stress tensor is given by

$$\sigma_{ij} = \lambda\, \delta_{ij}\, \varepsilon_{mm} + 2\mu\, \varepsilon_{ij},$$

where λ and μ are constants, δ_{ij} is the Kronecker delta tensor, and

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

where u_i is the displacement in the ith coordinate direction.

If

$$\frac{\partial \sigma_{ji}}{\partial x_j} + F_i = 0,$$

show that

$$(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})+\mu\nabla^2\mathbf{u}+\mathbf{F} = \mathbf{0}.$$

(8 marks)

(c) Let

$$\mathbf{F} = \phi\mathbf{a},$$

where ϕ is a scalar field and $\mathbf{a}$ is an arbitrary constant vector. By applying Gauss' theorem (see the formula sheet) to $\mathbf{F}$, show that

$$\int_V \nabla\phi\, dV = \int_S \phi\hat{\mathbf{n}}\, dS.$$

(6 marks)

End of Question Paper

Formula Sheet

Notation:

$$U(x_i,\, t_j) \equiv U_{i,j}$$

Forward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t}(x_i,\, t_j) \approx \frac{U_{i,\,j+1} - U_{i,\,j}}{\Delta t}$$

Forward difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x}(x_i,\, t_j) \approx \frac{U_{i+1,\,j} - U_{i,\,j}}{\Delta x}$$

Backward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t}(x_i,\, t_j) \approx \frac{U_{i,\,j} - U_{i,\,j-1}}{\Delta t}$$

Backward difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x}(x_i,\, t_j) \approx \frac{U_{i,j} - U_{i-1,j}}{\Delta x}$$

Central difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x}(x_i,\, t_j) \approx \frac{U_{i+1,\,j} - U_{i-1,\,j}}{2\Delta x}$$

Central difference formula for $\partial^2 U/\partial x^2$:

$$\frac{\partial^2 U}{\partial x^2}(x_i,\, t_j) \approx \frac{U_{i+1,\,j} - 2U_{i,\,j} + U_{i-1,\,j}}{(\Delta x)^2}$$

Relation between different parameters:

A number of relationships between E, ν, K, λ and μ hold and are summarized in Table 1. μ ($\equiv G$) is the elastic shear modulus, K the elastic bulk modulus, E the elastic stiffness (or Young's Modulus) and ν Poisson's ratio.

	E	ν	K	λ	$\mu \equiv G$
E, ν	-	-	$\frac{E}{3(1-2\nu)}$	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{E}{2(1+\nu)}$
E, K	-	$\frac{3K-E}{6K}$	-	$\frac{K(9K-3E)}{9K-E}$	$\frac{3KE}{9K-E}$
K, μ	$\frac{9\mu K}{3K+\mu}$	$\frac{3K-2\mu}{2(3K+\mu)}$	-	$K - \frac{2\mu}{3}$	-

Table 1: The relations between the properties of elastic bodies.

Gauss' theorem

Let $\mathbf{F}(x, y, z)$ be a vector field. Consider a closed volume V bounded by surface S. At each point on S let the outward normal be given by the unit vector $\hat{\mathbf{n}}$. Then *Gauss' theorem* (or the *divergence theorem*) is:

$$\int_V \nabla \cdot \mathbf{F}\, dV = \int_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS.$$