Data provided: Formula sheet

**MAS381**

The University Of Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS** — **Autumn Semester 2019–20**

**Mathematics III (Electrical)** — **2 hours**

*Attempt all the questions. The allocation of marks is shown in brackets.*

**1** (i) The function $F$ is defined by

$$F(a) = \oint_C \frac{4z^2 + z + 5}{z - a} dz,$$

where $C$ is the circle $|z| = 3$ taken in counter-clockwise direction. Use Cauchy's integral theorem to calculate $F(3.5), F(i)$ and $F(-1)$.

***(6 marks)***

(ii) Let us consider the function

$$f(z) = (\overline{z} + 1)^3 - 3\overline{z},$$

where $\overline{z}$ is the complex conjugate of $z$. Showing your working, determine whether this function satisfies the Cauchy-Riemann equations, or not.

***(8 marks)***

(iii) Determine the conjugate, $v(x, y)$, of the function

$$u(x, y) = e^x \cos y + e^y \cos x + xy,$$

where $u(x, y)$ is the real part of the complex function $f(z) = u(x, y) + iv(x, y)$. With the help of $u(x, y)$ and $v(x, y)$ determine the function $f(z)$ and find any constant in the expression of $f(z)$ by considering the condition $f(0) = 2 + 2i$. ***(11 marks)***

**2** (i) Find the Taylor series expansion of the function $\dfrac{1}{1 + 2z}$ about $z = i$. Show the region of convergence on the Argand diagram and indicate the pole that determines the radius of convergence. ***(10 marks)***

(ii) Find the Laurent series expansion of $\dfrac{1}{2i + (1 + 2i)z + z^2}$ in the region $1 < |z| < 2$. ***(15 marks)***

**3** (i) Find all the poles of $f(z) = \dfrac{z^4}{z^4 + 5z^2 + 4}$ and plot them on an Argand diagram. Hence evaluate the integral $\oint_C f(z)dz$, writing your solutions in the form $a + ib$, where $a$ and $b$ are real, where

(a) C is the circle $|z| = 3$

(b) C is the circle $|z + 3i/2| = 3/2$.

N.B. all integrations must be taken in the *counterclockwise* direction.

***(13 marks)***

(ii) The vector field $\mathbf{F}$ is given by

$$\mathbf{F} = (2x - y^2)\mathbf{i} + (x^2 + y^2 - z)\mathbf{j} + (2y + z^2)\mathbf{k}.$$

Evaluate the work $\int_C \mathbf{F} \cdot d\mathbf{r}$ done by the field $\mathbf{F}$ where $C$ is the half circle $x^2 + y^2 = 1$, $z = 0$ traversed in the counterclockwise direction starting from the point $(1, 0, 0)$ and finishing at $(-1, 0, 0)$

***(12 marks)***

**4** (i) Let $\mathbf{F}$ be the vector field

$$\mathbf{F} = 3y\mathbf{i} + 4x\mathbf{j} + 2z^2\mathbf{k}.$$

Let $C$ be the closed curve given by $x^2 + y^2 = 9$ and $z = 0$ and let $S$ be the surface of the open hemisphere $x^2 + y^2 + z^2 = 9$, $z \geq 0$. Let us consider $I$ be the line integral

$$I = \oint_C \mathbf{F} \cdot d\mathbf{r},$$

where integration is taken in the *counterclockwise* direction. Let us consider $J$ to be the surface integral

$$J = \iint_S curl(\mathbf{F}) \cdot d\mathbf{S}.$$

Evaluate $I$ and $J$ separately and verify that, in accordance with Stokes' Theorem, $I = J$. ***(19 marks)***

(ii) If $f(x, y, z) = x^3y^2 + \ln(y + z)$, calculate

$$\nabla^2 f + \nabla \cdot [\nabla \times (\nabla f)]$$

***(6 marks)***

## End of Question Paper

**Formula sheet**

- The general formula for the residue at a pole $z_0$, of order $m$ is

$$\frac{1}{(m-1)!} \lim_{z \to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[ (z-z_0)^m f(z) \right] \right\}$$

- Useful identities

$$\sin 2\theta = 2\sin\theta\cos\theta, \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

- The spherical area and volume elements are given by

$$dS = r^2 \sin\theta \; d\theta d\phi, \quad dV = r^2 \sin\phi \; dr \; d\phi \; d\theta$$

- The position vector in spherical coordinates for a unit radius is given as

$$\mathbf{r} = (x, y, z) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

- The coordinates between Cartesian and spherical coordinate systems transform according to

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

- The unit normal vector to the spherical surface is

$$\hat{\mathbf{n}} = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$