Data provided: Formula Sheet

MAS420

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2019–20**

Signal Processing **2 hours**

*Attempt **ALL** questions.*

Please leave this exam paper on your desk
Do not remove it from the hall

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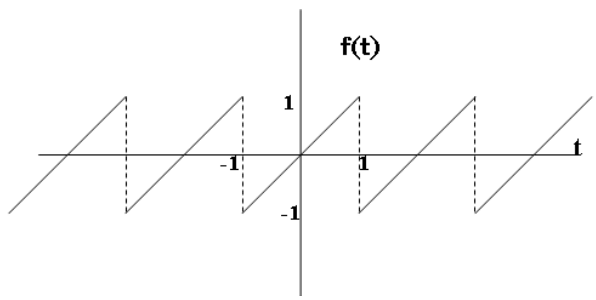
1 (i) In the Hilbert space of finite power signals of period T with inner product

$$(f, g) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)g^*(t)dt,$$

prove that the set $\phi_n(t) = e^{in\sigma t}$ for $-\infty < n < \infty$ and $\sigma = \dfrac{2\pi}{T}$, is an orthonormal set (both the orthogonality and unit norm conditions must be proved). ***(4 marks)***

(ii) Write down the period of the periodic signal shown in the figure below and find the complex Fourier coefficients for this signal. ***(7 marks)***



(iii) Use Parseval's theorem and your result in (ii) to derive the equality

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(5 marks)

(iv) The signal shown in (ii) is transmitted over a link which will not pass frequencies greater than 3π rad s^{-1}, but passes other frequencies unchanged. The received signal is $g(t)$. Find an expression for $g(t)$ as a sine/cosine series and also the percentage of power lost during transmission. ***(9 marks)***

2 (i) Define the equivalent rectangle resolution, τ, of a signal, $f(t)$, stating clearly the conditions under which it is defined. Sketch $|f(t)|^2$ against t for such a signal and indicate its equivalent rectangle resolution. ***(6 marks)***

(ii) The time-bandwidth theorem for a $\Omega-$bandlimited signal for which the equivalent rectangle resolution τ is defined states that $\Omega\tau \geq \pi$. Use the Cauchy-Schwarz inequality to prove the theorem and find the conditions needed to achieve the minimum possible time-bandwidth product.

(7 marks)

(iii) Using the property $\cos\omega_0 t = \dfrac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}$, show, with the aid of clear diagrams, that the signal

$$f(t) = \mathrm{sinc}(\alpha t)\cos(\omega_0 t),$$

where $\alpha > 0$ and $\omega_0 > 0$, has energy

$$E(f) = \begin{cases} \dfrac{\pi}{\alpha^2}\left(\alpha - \dfrac{\omega_0}{2}\right) & \text{if } \omega_0 < \alpha, \\ \dfrac{\pi}{2\alpha} & \text{if } \alpha \leq \omega_0. \end{cases}$$

Verify that $f(t)$ satisfies the conditions of the time-bandwidth theorem. In the case $\alpha < \omega_0$ calculate the bandwidth Ω and the equivalent rectangle resolution, τ, and hence verify that $\Omega\tau > \pi$. ***(12 marks)***

3 (i) Define the following:

- a linear shift-invariant (LSI) system;
- the system transfer function (STF), without any reference to the Fourier transform or the impulse response function;
- the impulse response function, without reference to the STF or convolution.

(4 marks)

(ii) With the aid of a clear diagram, explain what is meant by time domain and frequency domain processing for a LSI system, and why the two approaches are equivalent. ***(3 marks)***

(iii) A system uses integration to smooth noisy signals, i.e., if the input signal is $f(t)$, the output is given by

$$g(t) = \int_{t-T}^{t} f(s)ds,$$

where T is a constant. Show that this system is linear and shift-invariant and, using your definitions in part (i), verify that it has a STF given by

$$H(\omega) = Te^{-iT\omega/2}\,\mathrm{sinc}(T\omega/2).$$

(7 marks)

(iv) Working directly from your definition in part (i), find the impulse response function, $h(t)$, of the system and verify that $h(t) \longleftrightarrow H(\omega)$. ***(5 marks)***

(v) Use the STF to find the output from the system if the input is

$$f(t) = 1 - 3\sin\left(\frac{\pi t}{T}\right) + 2\cos\left(\frac{2\pi t}{T}\right),$$

simplifying your answer as much as possible. ***(6 marks)***

4 (i) From the Fourier transform pair $\overline{\delta}_T(t) \leftrightarrow \sigma\overline{\delta}_\sigma(\omega)$, where $\overline{\delta}_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$, $\overline{\delta}_\sigma(\omega)$ is defined similarly and $\sigma = 2\pi/T$, prove that

$$f_s(t) = f(t)\overline{\delta}_T(t) \leftrightarrow \frac{1}{T}\sum_{n=-\infty}^{\infty} F(\omega - n\sigma),$$

where $F(\omega)$ is the Fourier transform of $f(t)$ and $f_s(t)$ is the sampled signal of $f(t)$. ***(5 marks)***

(ii) Using the previous result, show that if $f(t)$ is Ω-bandlimited and $T < \pi/\Omega$, then $f(t)$ can be recovered exactly from the sampled signal, $f_s(t)$, by the sinc interpolation formula

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT)\,\mathrm{sinc}\left\{\frac{\sigma}{2}(t - nT)\right\}.$$

Clear diagrams are likely to help your answer. ***(6 marks)***

(iii) Find the Nyquist frequency, in Hz, of the signal

$$f(t) = \mathrm{sinc}(2\pi t).$$

(4 marks)

(iv) The signal from (iii) is sampled at 3/4 of its Nyquist frequency and the samples are used to form a signal $g(t)$ by sinc interpolation. Making use of clear diagrams, find $G(\omega)$ and hence $g(t)$. ***(7 marks)***

(v) Verify that $f(t) \neq g(t)$ by considering their respective values at $t = 1$. ***(3 marks)***

End of Question Paper

Formula sheet

Function Definitions:

Rectangular pulse:

$$p_a(t) = \begin{cases} 1 & |t| \le a \\ 0 & |t| > a \end{cases}$$

Triangular pulse:

$$q_a(t) = \begin{cases} 1 - |t|/a & |t| \le a \\ 0 & |t| > a \end{cases}$$

Step function:

$$U(t) = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases}$$

Fourier Transform Pairs:

$$\begin{aligned}
p_a(t) &\longleftrightarrow 2a\,\mathrm{sinc}(a\omega) \\
q_a(t) &\longleftrightarrow a\,\mathrm{sinc}^2(a\omega/2) \\
\mathrm{sinc}(at) &\longleftrightarrow \frac{\pi}{a} p_a(\omega) \\
\mathrm{sinc}^2(at) &\longleftrightarrow \frac{\pi}{a} q_{2a}(\omega) \\
\mathrm{e}^{-at}U(t) &\longleftrightarrow \frac{1}{a + i\omega} \\
\delta(t) &\longleftrightarrow 1 \\
\delta(t - t_0) &\longleftrightarrow \mathrm{e}^{-i\omega t_0} \\
1 &\longleftrightarrow 2\pi\delta(\omega) \\
\mathrm{e}^{i\omega_0 t} &\longleftrightarrow 2\pi\delta(\omega - \omega_0) \\
\mathrm{e}^{-t^2/2\sigma^2} &\longleftrightarrow \sigma\sqrt{2\pi}\mathrm{e}^{-\sigma^2\omega^2/2}
\end{aligned}$$

Fourier transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-i\omega t}\,\mathrm{d}t$$

Inverse Fourier transform:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\mathrm{e}^{i\omega t}\,\mathrm{d}\omega$$

Duality theorem: If $f(t) \longleftrightarrow F(\omega)$ then $F(t) \longleftrightarrow 2\pi f(-\omega)$

Scaling: If $f(t) \longleftrightarrow F(\omega)$ then $f(at) \longleftrightarrow \dfrac{1}{|a|}F(\omega/a)$.

Translation: If $f(t) \longleftrightarrow F(\omega)$ then $f(t - t_0) \longleftrightarrow \mathrm{e}^{-i\omega t_0}F(\omega)$.

Frequency Shift: If $f(t) \longleftrightarrow F(\omega)$ then $\mathrm{e}^{i\omega_0 t}f(t) \longleftrightarrow F(\omega - \omega_0)$

Fourier Series: If $f_T(t)$ is periodic with period T then, with $\sigma = 2\pi/T$, the complex Fourier series is

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{in\sigma t}$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t)\mathrm{e}^{-in\sigma t}\ \mathrm{d}t$$

Likewise, the real Fourier series is

$$f_T(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\sigma t + b_n \sin n\sigma t)$$

where

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f_T(t)\cos n\sigma t\ \mathrm{d}t$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f_T(t)\sin n\sigma t\ \mathrm{d}t$$

Parseval's Theorem: If V is a Hilbert space, $\{\phi_n\}$ is an orthonormal basis for V and $f = \sum_n c_n\phi_n$, then

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Plancherel's Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$\int_{-\infty}^{\infty} f(t)g^*(t)\ \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G^*(\omega)\ \mathrm{d}\omega$$

Energy Theorem: If $f(t) \longleftrightarrow F(\omega)$ then

$$\int_{-\infty}^{\infty} |f(t)|^2\ \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\ \mathrm{d}\omega$$

Convolution Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(s)g(t-s)\ \mathrm{d}s \longleftrightarrow F(\omega)G(\omega)$$

Product Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$f(t)g(t) \longleftrightarrow \frac{1}{2\pi}F(\omega) * G(\omega).$$