The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2019–20**

Analytic Number Theory **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Total mark is 80. Note that the questions do not carry equal marks: Q1 is worth 24 marks, Q2 is worth 33 marks and Q3 is worth 23 marks.

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1 (i) We will outline below yet another proof of the infinitude of primes. Let $N \geq 2$.

(a) Show that for any $n \in \mathbb{N}$, the number of positive integers less than or equal to N that are divisible by n^2 is $\left\lfloor \frac{N}{n^2} \right\rfloor$. ***(3 marks)***

(b) Recall that an integer $m \geq 2$ is *square-free* if it has a factorisation of the form $m = p_1 p_2 ... p_k$ for *distinct* primes $p_1, p_2, ..., p_k$ (i.e. m isn't divisible by any square number except 1). Define for $x \geq 2$ the function:

$$A(x) = \#\{m \leq x \mid m \text{ is square-free}\}.$$

Show that $N - A(N) \leq \sum_{n=2}^{\infty} \frac{N}{n^2}$. ***(2 marks)***

(c) Using the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, show that

$$A(N) \geq N\left(2 - \frac{\pi^2}{6}\right)$$

and hence prove that there are infinitely many primes. ***(6 marks)***

(ii) Let q be a prime and let $\Phi(x) = x^{q-1} + x^{q-2} + \ldots + x^2 + x + 1$ be the q-th cyclotomic polynomial.

(a) Let us call a prime p a "Φ-divisor" if there is an integer a such that p divides $\Phi(a)$. Show that there are infinitely many Φ-divisors.
(**Hint:** *You can follow a strategy inspired by Euclid's original proof of the infinite of primes. Assume that there are only finitely many Φ-divisors. Then try plugging into $\Phi(x)$ a certain product of the Φ-divisors to derive a contradiction.*) ***(4 marks)***

(b) Let p be another prime such that $p > q$. Assume that p divides $\Phi(a)$ for some integer a. Show that p divides $a^q - 1$, and deduce that the residue class of a in $(\mathbb{Z}/p\mathbb{Z})^\star$ has order q. ***(5 marks)***

(c) Deduce that there are infinitely many primes of the form $qn + 1$. ***(4 marks)***

2 (i) State Bertrand's Postulate and use it to show that $n!$ is not a square for any $n \geq 4$. ***(5 marks)***

(ii) Define the prime counting function $\pi(x)$ and state the Prime Number Theorem. ***(1 mark)***

(a) Evaluate $\lim_{x\to\infty} \frac{\pi(ax)}{\pi(x)}$ where a is a positive real number. ***(4 marks)***

(b) Deduce that there is a constant C such that for any $x > C$ we can find at least 2019 distinct primes between x and $2x$. ***(5 marks)***

(iii) Now let $\sigma_0(n)$ be the number of (positive) divisors of n. You may assume that $\sigma_0(n), \sigma_0(n)^3$ are multiplicative functions.

(a) Write down the Euler product expansion of the Riemann zeta function $\zeta(s)$. Show that $\sum_{n=1}^{\infty} \frac{\sigma_0(n)}{n^s} = \zeta(s)^2$.
(**Hint:** *Recall the unit function u defined by $u(n) = 1$ for all n. What does $u \star u$ look like?*) ***(4 marks)***

(b) By considering the divisors of p^k, derive a formula for $\sigma_0(p^k)$ if p is a prime number and $k \geq 1$. ***(2 marks)***

(c) Give an indication of how you would derive the series expansion

$$\frac{1 + 4x + x^2}{(1-x)^4} = 1 + 2^3x + 3^3x^2 + 4^3x^3 + \ldots$$

given the identity

$$\frac{1+x}{(1-x)^3} = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \ldots = \sum_{n=0}^{\infty}(n+1)^2x^n.$$

You do not need to carry out any calculation. ***(2 marks)***

(d) Deduce, using the preceding parts, that

$$\sum_{n=1}^{\infty} \frac{\sigma_0(n)^3}{n^s} = \zeta(s)^4 \prod_{\text{primes } p} \left(1 + \frac{4}{p^s} + \frac{1}{p^{2s}}\right).$$

(4 marks)

(e) Prove that

$$\sum_{m|n} \sigma_0(m)^3 = \Big(\sum_{m|n} \sigma_0(m)\Big)^2$$

for any positive integer n.
(**Hint:** *You might like to reduce the problem to the case when n is a power of a prime. If required you can use, without proof, the identity* $1^3 + 2^3 + \ldots + N^3 = (1 + 2 + \ldots + N)^2$.) ***(6 marks)***

3 This question asks you to illustrate the proof of Dirichlet's Theorem in a specific case.

(i) List the characters of $(\mathbb{Z}/10\mathbb{Z})^\star$. ***(5 marks)***

(ii) Prove that $L(1,\chi) \neq 0$ for each non-trivial character χ on your list. ***(7 marks)***

(iii) Using the character table, show that for a prime p, we have

$$\sum_\chi \chi(\overline{7})^{-1}\chi(\overline{p}) = \begin{cases} 4 & \text{if } p \equiv 7 \mod 10 \\ 0 & \text{otherwise} \end{cases}$$

where the sum runs over the characters of $(\mathbb{Z}/10\mathbb{Z})^\star$. ***(2 marks)***

(iv) Prove that there are infinitely many primes congruent to 7 modulo 10.

(*You may assume that for any character* χ*, the sum* $\sum_{p\neq 2,5}\sum_{n=2}^{\infty}\frac{\chi(p)^n}{np^{ns}}$ *converges to a finite limit as* $s \to 1^+$*. Do state the analytic properties of the* $L(s,\chi)$*'s that are relevant to the proof, but you do not need to prove them.*) ***(9 marks)***

End of Question Paper