The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2019-20**

Fields **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) State the Subfield Criterion. ***(3 marks)***

(ii) Prove that the intersection $K_1 \cap K_2$ of two subfields K_1 and K_2 of a field L is also a subfield of L. ***(2 marks)***

(iii) For each of the subsets J_1 and J_2 of $\mathbb{C}$ specified below determine, with justification, whether it is a subfield of $\mathbb{C}$:

(a) $J_1 = \{a + b\sqrt{-2} : \ a, b \in \mathbb{Q}\}$. ***(5 marks)***

(b) $J_2 = \{a + b\sqrt{3} + c\sqrt{5} : \ a, b, c \in \mathbb{Q}\}$. ***(5 marks)***

(iv) Show that the following subfields L_1 and L_2 of $\mathbb{R}$ are equal where

$$L_1 = \mathbb{Q}(\sqrt{2}, \sqrt{3}) \ \text{ and } \ L_2 = \mathbb{Q}\left(\sqrt{2} + \sqrt{3}, \sqrt{2} - \sqrt{3}, \frac{5\sqrt{2} + 7\sqrt{3}}{2\sqrt{2} + 3\sqrt{3}}\right).$$

(5 marks)

(v) Is the element $a = 2\sqrt{2} - 3\sqrt{3} + 1$ equal to 0? Justify your response. ***(5 marks)***

2 (i) State Gauss' Lemma. ***(2 marks)***

(ii) Give a definition of the content $c(f)$ of a polynomial $f \in \mathbb{Z}[x]$ and prove Gauss' Lemma. ***(6 marks)***

(iii) Let $K \subseteq L$ be a field extension and $a \in L$ be an algebraic element over the field K.

(a) Define the *minimal polynomial* $m(x) \in K[x]$ of the element a over K and prove that the polynomial $m(x)$ irreducible over K. ***(6 marks)***

(b) Suppose that $n = \deg(m(x))$. Prove that the set of elements $1, a, a^2, \ldots, a^{n-1}$ form a basis for the vector space $K(a)$ over the field K. ***(8 marks)***

(c) Find the minimal polynomial $m(x) \in \mathbb{Q}(\sqrt{3})[x]$ of the element $a = \sqrt{3} + 3\sqrt{2}$ over the field $\mathbb{Q}(\sqrt{3})$. ***(3 marks)***

3 (i) (a) State Eisenstein's Irreducibility Criterion. ***(2 marks)***

(b) Prove Eisenstein's Irreducibility Criterion. ***(8 marks)***

(c) Show that the polynomial $13x^{10} - 6x^7 - 36x^3 + 12$ is irreducible in $\mathbb{Q}[x]$. ***(2 marks)***

(ii) (a) State the 'Shifted Eisenstein's Irreducibility Criterion'. ***(3 marks)***

(b) Prove the 'Shifted Eisenstein's Irreducibility Criterion'. ***(7 marks)***

(c) Show that the polynomial $f(x) = x^2 + 7x - 5$ is irreducible in $\mathbb{Q}[x]$ by using the 'Shifted Eisenstein's Irreducibility Criterion' (or otherwise). ***(3 marks)***

4 (i) Let σ be an automorphism of a field L and $L^{\sigma} = \{a \in L \mid \sigma(a) = a\}$. Prove that L^{σ} is a subfield of L. ***(4 marks)***

(ii) Let $L = \mathbb{C}(x, y)$ be the field of rational functions in two variables x and y over the field of complex numbers $\mathbb{C}$ and σ is a $\mathbb{C}$-automorphism of the field L given by the rule $\sigma(x) = -x$ and $\sigma(y) = e^{\frac{2\pi i}{n}} y$ for some $n \geq 2$ (where $i = \sqrt{-1}$).

(a) Define the order $\text{or}(\sigma)$ of the automorphism σ. ***(2 marks)***

(b) Find the order $\text{or}(\sigma)$ of the automorphism σ. ***(6 marks)***

(c) Suppose that $n = 3$. Let $K = \mathbb{C}(x^2, y^3)$. Prove that $K \subseteq L^{\sigma}$. ***(3 marks)***

(d) Suppose that $n = 3$. Prove that $x \notin K$ and $y \notin K$. ***(4 marks)***

(e) Suppose that $n = 3$. Show that $p(t) = t^2 - x^2 \in K[t]$ is the minimal polynomial of the element x over K. Show that $q(t) = t^3 - y^3 \in K[t]$ is the minimal polynomial of the element y over K. ***(6 marks)***

End of Question Paper