



The
University
Of
Sheffield.

MAS462

SCHOOL OF MATHEMATICS AND STATISTICS

**Autumn Semester
2019–20**

Financial Mathematics

2 hours and 30 minutes

Attempt all the questions. The allocation of marks is shown in brackets.

**Please leave this exam paper on your desk
Do not remove it from the hall**

Registration number from U-Card (9 digits)
to be completed by student

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- 1 (i) Consider the following five bonds with face value of £100:

Time to maturity (in years)	Annual interest (paid every 6 months)	Bond price (in £)
0.25	0	99.0
0.5	0	97.8
1.0	0	95.5
1.5	8%	104.5
2.0	12%	113.0

- (a) Find the 0.25, 0.5 and 1-year spot interest rates. *(2 marks)*
- (b) Use the bootstrap method to find the 1.5 and 2-year spot interest rates. *(5 marks)*
- (ii) Consider a forward contract on foreign currency. By considering two portfolios involving positions on the forward contract and domestic and foreign currency deposits show that

$$Fe^{-rT} = Se^{-r_f T},$$

where F is the forward exchange rate (i.e., domestic currency paid for 1 unit of foreign currency paid on delivery), S is the spot price of the asset (i.e., domestic currency paid at the present for 1 unit of foreign currency), T is the time to maturity of the forward contract (in years), r is the T -year spot interest rate for domestic currency and r_f is the T -year spot interest rate for the foreign currency. *(8 marks)*

- (iii) Suppose that the domestic one-year and three-year spot interest rates are 4% and 5% respectively.
- (a) What is the forward rate for the period starting in one year and ending in three years? *(1 mark)*
- (b) Suppose that you are offered by a risk free institution the opportunity to deposit or borrow £10,000,000 in one year for a period of two years earning an interest rate of 5%. Describe in detail an arbitrage opportunity available to you. *(9 marks)*

- 2 (i) Consider two European call options with the same underlying asset and same expiration date in T -years, and with strike prices $X_1 < X_2$. Consider also a portfolio which is long one call option with strike price X_1 and short one call option with strike X_2 . Let r be the T -year spot interest rate.

Sketch the payoff function of the portfolio. *(5 marks)*

- (ii) Let c be the price of a European call option on stock which pays no dividends; let X be the strike price of the option, let S be the spot price of the stock, let T be the time to the option expiration (in years) and let r be the T -year spot interest rate. By comparing a portfolio consisting of options and cash, to a portfolio consisting of one share, prove that

$$c \geq S - Xe^{-rT}. \quad (4 \text{ marks})$$

- (iii) Let c and p be the prices of a European call option and put option, respectively, on stock which pays no dividends; let X be the strike price of both options, let S be the spot price of the stock, let T be the time to the options expiration (in years) and let r be the T -year spot interest rate. By comparing two portfolios consisting of options and cash, prove that

$$c + Xe^{-rT} = p + S. \quad (10 \text{ marks})$$

- (iv) Suppose that c_1, c_2 , and c_3 are the prices of European call options with strike prices X_1, X_2 , and X_3 respectively, where $X_3 > X_2 > X_1$ and $X_3 - X_2 = X_2 - X_1$. We are told that all options have the same expiration date. Show that

$$c_2 \leq \frac{1}{2}(c_1 + c_3). \quad (6 \text{ marks})$$

- 3** (i) Explain the principle of risk-neutral valuation. *(3 marks)*
- (ii) The price of a stock which pays no dividends is currently £20. Over each of the next three 1-year periods the stock price will either increase by 10% or decrease by 10%. Suppose that all interest rates are constant and equal to 3%.
- (a) Use a binomial tree to find the price of a three-year American put option on this stock with strike price £20. *(11 marks)*
- (b) Describe all circumstances when a rational investor should exercise the option. *(3 marks)*
- (iii) Consider a European call option on an asset with spot price S , expiration time in one year, and strike price S . You are told that a year from now the price of the asset will have either risen by a factor of u ($u > 1$) or will have decreased by a factor of $\frac{1}{u}$. If the interest rate on one-year deposits is $r > 0$, show that the price of the European call option is an increasing function of the variable u . *(8 marks)*

4 (i) State Ito's Lemma. (5 marks)

(ii) In this question we consider a European call option and a European put option, both with the same underlying stock, the same strike price X and the same maturity time $T > 0$. For any time $0 \leq t \leq T$, let $S_t = S$ denote the spot price of the underlying stock, let $c(S, t)$ be the price of European call and let $p(S, t)$ be the price of European put option at time t . Assume that all spot interest rates are constant and equal to r . Assume also that the underlying stock price follows the Ito process

$$dS = rSdt + \sigma SdB$$

(a) Verify that $f(S, t) = Xe^{-r(T-t)} - S$ is a solution of the Black-Scholes partial differential equation

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf.$$

(2 marks)

(b) Deduce that $g(S, t) = Xe^{-r(T-t)} - S + c(S, t) - p(S, t)$ is also a solution of the Black-Scholes partial differential equation. (3 marks)

(c) By considering the case $t = T$, show that (b) implies that

$$c(S, t) + Xe^{-r(T-t)} = p(S, t) + S. \quad (6 \text{ marks})$$

(iii) For any time $0 \leq t \leq T$, let $S_t = S$ denote the spot price of a stock. Assume that all spot interest rates are constant and equal to r . We also assume that the stock price S_t follows geometric Brownian motion $dS = rSdt + \sigma SdB$.

(a) Show that the process $S_t^n, n \geq 2$ also follows geometric Brownian motion, and find its drift. (3 marks)

(b) Now consider a derivative, on the same stock, that pays off S_T^n at time T , where S_T is the stock price at that time. If you are told that the price of the derivative at time t ($t \leq T$) has the form

$$h(t)S_t^n,$$

where S_t is the stock price at time t , h is a function only of t , and that the price of the derivative satisfies the Black-Scholes equation,

(α) find the differential equation that $h(t)$ satisfies. (2 marks)

(β) find the value of $h(T)$, which plays the role of a "boundary" condition, for the above differential equation (2 marks)

(γ) show that

$$h(t) = e^{[\frac{1}{2}\sigma^2 n(n-1) + r(n-1)](T-t)},$$

where r is the risk-free interest rate and σ is the stock price volatility. (2 marks)

End of Question Paper