



SCHOOL OF MATHEMATICS AND STATISTICS

Autumn Semester
2019-20

Fields

2 hours 30 minutes

Attempt all the questions. The allocation of marks is shown in brackets.

- 1 (i) State the Subfield Criterion. (3 marks)
- (ii) Prove that the intersection $K_1 \cap K_2$ of two subfields K_1 and K_2 of a field L is also a subfield of L . (2 marks)
- (iii) For each of the subsets J_1 and J_2 of \mathbb{C} specified below determine, with justification, whether it is a subfield of \mathbb{C} :
- (a) $J_1 = \{a + b\sqrt{-2} : a, b \in \mathbb{Q}\}$. (5 marks)
- (b) $J_2 = \{a + b\sqrt{3} + c\sqrt{5} : a, b, c \in \mathbb{Q}\}$. (5 marks)
- (iv) Show that the following subfields L_1 and L_2 of \mathbb{R} are equal where
- $$L_1 = \mathbb{Q}(\sqrt{2}, \sqrt{3}) \text{ and } L_2 = \mathbb{Q}\left(\sqrt{2} + \sqrt{3}, \sqrt{2} - \sqrt{3}, \frac{5\sqrt{2} + 7\sqrt{3}}{2\sqrt{2} + 3\sqrt{3}}\right).$$
- (5 marks)
- (v) Is the element $\alpha = 2\sqrt{2} - 3\sqrt{3} + 1$ equal to 0? Justify your response. (5 marks)

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- (i) State Gauss' Lemma. *(2 marks)*
 - (ii) Give a definition of the content $c(f)$ of a polynomial $f \in \mathbb{Z}[x]$ and prove Gauss' Lemma. *(6 marks)*
 - (iii) Let $K \subseteq L$ be a field extension and $\alpha \in L$ be an algebraic element over the field K .
 - (a) Define the *minimal polynomial* $m(x) \in K[x]$ of the element α over K and prove that the polynomial $m(x)$ irreducible over K . *(6 marks)*
 - (b) Suppose that $n = \deg(m(x))$. Prove that the set of elements $1, \alpha, \alpha^2, \dots, \alpha^{n-1}$ form a basis for the vector space $K(\alpha)$ over the field K . *(8 marks)*
 - (c) Find the minimal polynomial $m(x) \in \mathbb{Q}(\sqrt{3})[x]$ of the element $\alpha = \sqrt{3} + 3\sqrt{2}$ over the field $\mathbb{Q}(\sqrt{3})$. *(3 marks)*
- 3**
- (i)
 - (a) State Eisenstein's Irreducibility Criterion. *(2 marks)*
 - (b) Prove Eisenstein's Irreducibility Criterion. *(8 marks)*
 - (c) Show that the polynomial $13x^{10} - 6x^7 - 36x^3 + 12$ is irreducible in $\mathbb{Q}[x]$. *(2 marks)*
 - (ii)
 - (a) State the 'Shifted Eisenstein's Irreducibility Criterion'. *(3 marks)*
 - (b) Prove the 'Shifted Eisenstein's Irreducibility Criterion'. *(7 marks)*
 - (c) Show that the polynomial $f(x) = x^2 + 7x - 5$ is irreducible in $\mathbb{Q}[x]$ by using the 'Shifted Eisenstein's Irreducibility Criterion' (or otherwise). *(3 marks)*

- 4 (i) Let σ be an automorphism of a field L and $L^\sigma = \{a \in L \mid \sigma(a) = a\}$. Prove that L^σ is a subfield of L . *(4 marks)*
- (ii) Let $L = \mathbb{C}(x, y)$ be the field of rational functions in two variables x and y over the field of complex numbers \mathbb{C} and σ is a \mathbb{C} -automorphism of the field L given by the rule $\sigma(x) = -x$ and $\sigma(y) = e^{\frac{2\pi i}{n}}y$ for some $n \geq 2$ (where $i = \sqrt{-1}$).
- (a) Define the order $\text{or}(\sigma)$ of the automorphism σ . *(2 marks)*
- (b) Find the order $\text{or}(\sigma)$ of the automorphism σ . *(6 marks)*
- (c) Suppose that $n = 3$. Let $K = \mathbb{C}(x^2, y^3)$. Prove that $K \subseteq L^\sigma$. *(3 marks)*
- (d) Suppose that $n = 3$. Prove that $x \notin K$ and $y \notin K$. *(4 marks)*
- (e) Suppose that $n = 3$. Show that $p(t) = t^2 - x^2 \in K[t]$ is the minimal polynomial of the element x over K . Show that $q(t) = t^3 - y^3 \in K[t]$ is the minimal polynomial of the element y over K . *(6 marks)*

End of Question Paper