The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2019**

Analysis II **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 (i) Define the concept of metric space. **(5 marks)**

(ii) Define $d_3\colon\ \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$d_3(\underset{\sim}{x}, \underset{\sim}{y}) = |x^1 - y^1| + 3|x^2 - y^2|,$$

where $\underset{\sim}{x} = (x^1, x^2)$, $\underset{\sim}{y} = (y^1, y^2)$.

(a) Show that d_3 is a metric on $\mathbb{R}^2$.

(b) Show that

$$d_1(\underset{\sim}{x}, \underset{\sim}{y}) \leqslant d_3(\underset{\sim}{x}, \underset{\sim}{y}) \leqslant 3d_1(\underset{\sim}{x}, \underset{\sim}{y})$$

for all $\underset{\sim}{x}, \underset{\sim}{y} \in \mathbb{R}^2$. Here

$$d_1(\underset{\sim}{x}, \underset{\sim}{y}) = |x^1 - y^1| + |x^2 - y^2|,$$

as usual.

(6 marks)

(iii) Let (X, d) be a metric space.

(a) Prove that, for all $x, y, z \in X$,

$$|d(x, z) - d(y, z)| \leqslant d(x, y).$$

(b) Prove that, for all $x, y, a, b \in X$,

$$|d(x, y) - d(a, b)| \leqslant d(x, a) + d(y, b)$$

(6 marks)

(iv) Define what it means for a sequence (x_n) in a metric space (X, d) to *converge to an element* $x \in X$. **(2 marks)**

(v) Let (x_n) and (y_n) be two sequences in a metric space (X, d).

(a) Show that if $x_n \to a$ and $y_n \to a$ then $d(x_n, y_n) \to 0$.

(b) Show that if $x_n \to a$ and $y_n \to b$ then $d(x_n, y_n) \to d(a, b)$.

(6 marks)

2 (i) Let $f\colon X \to Y$ be a function from a metric space (X, d_X) to a metric space (Y, d_Y). Define what it means for f to be $\varepsilon\delta$-*continuous at* $x \in X$, and what it means for f to be *sequentially-continuous at* $x \in X$.

(6 marks)

(ii) Now consider a continuous function $h\colon [a, b] \to \mathbb{R}$ where $[a, b]$ is a closed bounded interval in $\mathbb{R}$. Suppose that $h(c) > 0$ for a certain $c \in (a, b)$. Show that there exists $A > 0$ and $\delta > 0$ such that

$$h(x) > \tfrac{1}{2}A \text{ for all } c - \delta < x < c + \delta.$$

(4 marks)

(iii) Define a function $d_1\colon C[a, b] \times C[a, b] \to \mathbb{R}$ by

$$d_1(f, g) = \int_a^b |f(x) - g(x)|\, dx.$$

Prove that d_1 is a metric on $C[a, b]$. **(10 marks)**

(iv) In the metric space $(C[0, 1], d_1)$, let U be the set

$$U = \{f \ : \ f(1) = 1\}.$$

Determine whether or not U is closed, giving reasons for your answer.

(5 marks)

3 (i) Let (X, d) be a metric space.

(a) Define what it means for a sequence (x_n) to be a *Cauchy sequence*.

(b) Define what it means for (X, d) to be *complete*.

(c) Show that a convergent sequence is Cauchy. **(7 marks)**

(ii) Consider $C[0,1]$ with the metric d_∞ defined by

$$d_\infty(f, g) = \sup\{|f(x) - g(x)| \ : \ x \in [0,1]\}.$$

Define (f_n), for $n \in \mathbb{N}$, by

$$f_n(x) = 1 + \frac{x}{2} + \frac{x^2}{2^2} + \cdots + \frac{x^n}{2^n},$$

for $x \in [0,1]$.

(a) For $m > n$, show that $f_m(x) - f_n(x) \geqslant 0$ for all $x \in [0,1]$.

Show also that $f_m - f_n$ is strictly increasing on $(0,1]$.

Hence, or otherwise, show that, for $m > n$,

$$d_\infty(f_n, f_m) = f_m(1) - f_n(1) = \frac{1}{2^n} - \frac{1}{2^m}.$$

You may use, if you wish, the formula

$$1 + r + r^2 + \cdots + r^k = \frac{1 - r^{k+1}}{1 - r}, \qquad r \neq 1.$$

(b) Show that (f_n) is a Cauchy sequence in $(C[0,1], d_\infty)$.

(18 marks)

4 (i) Let $X = [0, 2\pi) \subseteq \mathbb{R}$ have the subspace metric from the usual metric on $\mathbb{R}$ and let $Y = \{(x^1, x^2) \in \mathbb{R}^2 \ : \ (x^1)^2 + (x^2)^2 = 1\}$ have the subspace metric from the usual Euclidean metric d_2.

You may assume that $f\colon\ X \to Y$, $f(t) = (\cos(t), \sin(t))$ is continuous and a bijection.

Show that the inverse function $f^{-1}\colon\ Y \to X$ is not continuous.

(4 marks)

(ii) (a) Let (X, d_X) be a metric space. Define what it means for a subset $A \subseteq X$ to be *compact*.

(b) Prove that a compact subset of a metric space is closed.

(c) Prove that a closed subset of a compact metric space is compact.

(9 marks)

(iii) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f\colon\ X \to Y$ be a continuous function.

(a) Prove that if $A \subseteq X$ is compact, then $f(A) \subseteq Y$ is compact.

(b) Now assume that X is compact and that $f\colon\ X \to Y$ is a continuous bijection.

Prove that the inverse function $f^{-1}\colon\ Y \to X$ is continuous.

(12 marks)

[You should state clearly, but need not prove, any result(s) from the course that you use.]

End of Question Paper