The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2019–20**

Topics in Advanced Fluid Mechanics **2 hours 30 minutes**

Marks will be awarded for your best ***four*** *answers.*

1 Consider the 3D Navier-Stokes equations (written with standard notations) for an incompressible fluid of a constant unit density in the whole space $\mathbb{R}^3$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \nu \nabla^2 \boldsymbol{u},$$

$$\nabla \cdot \boldsymbol{u} = 0.$$

We assume the fluid is at rest at infinity and spatial averages of $\boldsymbol{u}$ and p are zero.

(i) Derive a set of equations for the impulse defined by $\boldsymbol{\gamma} = \boldsymbol{u} + \nabla\phi$,

$$\frac{D\boldsymbol{\gamma}}{Dt} = -(\nabla \boldsymbol{u})^T \boldsymbol{\gamma} + \nabla\lambda + \nu\nabla^2\boldsymbol{\gamma},$$

$$\frac{D\phi}{Dt} = p - \frac{|\boldsymbol{u}|^2}{2} + \lambda + \nu\nabla^2\phi,$$

where λ is an arbitrary scalar function of $\boldsymbol{x}$ and t. ***(15 marks)***

(ii) In the inviscid case $\nu = 0$, by using a special choice of $\lambda = 0$ (geometric gauge) show that the function

$$f(t) = \int_{\mathbb{R}^3} \phi \, d\boldsymbol{x}$$

satisfies

$$\frac{df}{dt} = -\int_{\mathbb{R}^3} \frac{|\boldsymbol{u}|^2}{2} \, d\boldsymbol{x}.$$

Calculate $\displaystyle\frac{d}{dt}\int_{\mathbb{R}^3} \frac{|\boldsymbol{u}|^2}{2} \, d\boldsymbol{x}$ and on this basis solve the above equation for $f(t)$ in terms of $f(0)$.

(10 marks)

2 We consider the Burgers equation in $\mathbb{R}^1$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}. \tag{1}$$

For $t > 0$, we consider a solution of the following form

$$u(x,t) = \frac{1}{\sqrt{t}}U(X), \quad \text{where} \quad X = \frac{x}{\sqrt{t}}$$

under the boundary conditions $|U(X)|, |U'(X)| \to 0$ as $|X| \to \infty$. We determine the functional form of $U(X)$ in the following steps.

(i) By direct computations show that $\dfrac{\partial X}{\partial t} = -\dfrac{1}{2t}X$ and

$$\frac{\partial u}{\partial t} = -\frac{1}{2}t^{-3/2}(U + XU'),\ \frac{\partial u}{\partial x} = t^{-1}U',\ \frac{\partial^2 u}{\partial x^2} = t^{-3/2}U''.$$

(8 marks)

(ii) From (1) derive

$$U'' + (U + XU') = 2UU',$$

and further reduce it to

$$U' + XU = U^2. \tag{2}$$

(8 marks)

(iii) Using $U' + XU = e^{-X^2/2}(Ue^{X^2/2})'$ in (2) and introducing a set of variables

$$V = Ue^{X^2/2},\ Y = \int_0^X e^{-z^2/2}dz,$$

reduce (2) to

$$\frac{dV}{dY} = V^2. \tag{3}$$

(4 marks)

(iv) Solving (3), derive

$$U(X) = \frac{Ce^{-X^2/2}}{1 - C\int_0^X e^{-z^2/2}dz},$$

where C is a constant. Rewrite this expression in the form of the Cole-Hopf transformation. ***(5 marks)***

3 Consider a model equation for vorticity ω defined in $\mathbb{R}^1$:

$$\frac{\partial \omega}{\partial t} = \omega H[\omega], \tag{1}$$

with an initial condition

$$\omega(x, t = 0) = \omega_0(x).$$

Here $H[\omega]$ denotes the Hilbert transform on $\mathbb{R}^1$

$$H[\omega](x) = \frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{\omega(y)}{x - y} dy,$$

where $⨍$ denotes a principal-value integral.

You may assume the following formulas

$$H\left[\frac{a}{x^2 + a^2}\right] = \frac{x}{x^2 + a^2}, \; H\left[\frac{x}{x^2 + a^2}\right] = -\frac{a}{x^2 + a^2},$$

where a is a constant.

(i) Show that the following form of a solution

$$\omega(x, t) = \frac{a(t)}{x^2 + a(t)^2}$$

is inconsistent with (1), that is, $a(t) \equiv 0$ is the only possibility of solution. ***(6 marks)***

(ii) Show that the following form of a solution

$$\omega(x, t) = \frac{x}{x^2 + a(t)^2}$$

is consistent with (1) by determining $a(t)$ in terms of $a_0 = a(0)$. ***(6 marks)***

(iii) For the solution in (ii), determine $\max_x \omega(t)$ and $\max_x H[\omega](t)$ and the distance between these maxima. Assume that $a(0) < 0$. ***(7 marks)***

(iv) Sketch graphs of $\omega(x, t)$ and $H[\omega](x, t)$ at some positive $t > 0$. ***(6 marks)***

4 We consider the Navier-Stokes equations subject to an external straining flow. The velocity field reads

$$\boldsymbol{u} = -ax\boldsymbol{i} + v(x,t)\boldsymbol{j} + az\boldsymbol{k}, \quad (a > 0),$$

where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ denote the standard Cartesian unit vectors.

(i) Derive

$$\frac{\partial \omega}{\partial t} = a\frac{\partial}{\partial x}(x\omega) + \nu\frac{\partial^2 \omega}{\partial x^2} \tag{1}$$

from the vorticity equations

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} + \nu\nabla^2\boldsymbol{\omega}.$$

Here the vorticity field is given by $\boldsymbol{\omega} = \omega(x,t)\boldsymbol{k}$, where $\omega(x,t) = \dfrac{\partial v}{\partial x}$,

(7 marks)

(ii) We consider (1) subject to the boundary conditions $|\omega(x,t)|, \left|\dfrac{\partial \omega}{\partial x}\right| \to 0$ sufficiently fast, as $|x| \to \infty$. Show that $M = \displaystyle\int_{-\infty}^{\infty} \omega(x,t)dx$ is independent of time, that is, $\dfrac{dM}{dt} = 0$. ***(6 marks)***

(iii) Show that a steady solution of (1) is given by

$$\omega(x) = C\exp\left(-\frac{a}{2\nu}x^2\right), \tag{2}$$

where C is a constant defined by $C = \sqrt{\dfrac{a}{2\pi\nu}}M$. ***(6 marks)***

Hint: $\displaystyle\int_{-\infty}^{\infty} e^{-\alpha x^2}dx = \sqrt{\frac{\pi}{\alpha}} \;\; (\alpha > 0).$

(iv) The energy dissipation rate, per unit length along the y-axis, is defined by

$$E = \nu\int_{-\infty}^{\infty} \omega(x)^2 dx.$$

Compute E for the above solution (2) and show that it vanishes in the limit of $\nu \to 0$. ***(6 marks)***

5 The motion of a vortex layer of uniform strength, under periodic boundary conditions, is governed by

$$\frac{\partial z(\alpha,t)^*}{\partial t} = -\frac{i}{4\pi} ⨍_0^{2\pi} \cot \frac{z(\alpha,t)-z(\beta,t)}{2} d\beta, \qquad (1)$$

where $z(\alpha,t) = x(\alpha,t) + iy(\alpha,t)$ denotes the position of a fluid particle α on the layer, * is complex conjugate and $⨍$ is a principal-value integral. We shall study the stability of an initially flat state $z_0(\alpha) = \alpha$.

(i) Setting $z(\alpha) = \alpha + i\epsilon f(\alpha,t)$, where $f(\alpha,0) = 0$, derive from (1)

$$\epsilon \frac{\partial f(\alpha)^*}{\partial t} = \frac{1}{4\pi} ⨍_0^{2\pi} \cot \left(\frac{\alpha - \beta + i\epsilon\{f(\alpha) - f(\beta)\}}{2} \right) d\beta \qquad (2)$$

(4 marks)

(ii) Setting $F(\epsilon) = \cot \dfrac{\alpha - \beta + i\epsilon\{f(\alpha) - f(\beta)\}}{2}$, show that

$$F'(0) = -i\{f(\alpha) - f(\beta)\} \frac{\partial}{\partial \beta} \cot \frac{\alpha - \beta}{2}.$$

Using $F(\epsilon) \approx F(0) + \epsilon F'(0)$ for small ϵ, from (2) derive a linearised equation

$$\frac{\partial f(\alpha)^*}{\partial t} = -\frac{i}{4\pi} ⨍_0^{2\pi} \{f(\alpha) - f(\beta)\} \frac{\partial}{\partial \beta} \cot \frac{\alpha - \beta}{2} d\beta. \qquad (3)$$

Hint: $(\cot x)' = -(\operatorname{cosec} x)^2$. ***(6 marks)***

(iii) Using integration by parts, rewrite (3) as

$$\frac{\partial f^*}{\partial t} = -\frac{i}{2} H \left[\frac{\partial f}{\partial \beta} \right], \qquad (4)$$

where the Hilbert transform under periodic boundaries is defined by

$$H[f] = \frac{1}{2\pi} ⨍_0^{2\pi} f(\beta) \cot \frac{\alpha - \beta}{2} d\beta.$$

Hint: Properties of the Hilbert transform we have learned for $\mathbb{R}^1$ remain valid for periodic boundary conditions. ***(4 marks)***

(iv) Derive from (4)

$$\frac{\partial^2 f}{\partial t^2} = -\frac{1}{4} \frac{\partial^2 f}{\partial \alpha^2}.$$

(6 marks)

(v) For a perturbation of the form

$$f \propto \exp(ik\alpha + \lambda t),$$

find the growth rate λ as a function of k and hence determine whether the sheet is linearly stable or not. ***(5 marks)***

End of Question Paper