The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS Autumn Semester 2019–20

WAVES 2 hours

Marks will be awarded for your best FOUR *answers. The marks awarded to each question or section of question are shown in italics.*

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1 The transverse displacement of a stretched string is $u(x,t)$, where x is the distance along the string and $t \geq 0$ is time.

(i) *Verify* that the d'Alembert general solution

$$u(x,t) = f(x - ct) + g(x + ct),$$

where f and g are arbitrary functions and c is a constant, satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{1}$$

Hence find $u(x,t)$, for $t > 0$, of an infinite stretched string that, at $t = 0$, is at rest with displacement $\sin(x)$.

(10 marks)

(ii) Next, assume that a finite string is held fixed at its endpoints $x = 0$ and $x = L$ and the transverse displacement $u(x,t)$ satisfies the wave equation given by (1). At $t = 0$ the displacement is zero. *Verify* that

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(\alpha_n x) \sin(\alpha_n ct)$$

satisfies equation (1) and the initial and boundary conditions, where B_n and α_n $(n = 1, 2, 3, \ldots)$ are constants. Determine α_n $(n = 1, 2, 3, \ldots)$.

If it is further given that, at $t = 0$, the velocity $\partial u/\partial t = f(x)$, find an integral formula for B_n.

(15 marks)

2 A uniform finite string of length L and density ρ undergoes small transverse vibrations with displacement $y(x,t)$, where $y_{tt} = c^2 y_{xx}$, and c^2 is a constant.

(i) Given that $y(0,t) = y(L,t) = 0$, *derive* by using the method of separation of variables that the general solution is

$$y(x,t) = \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right) \right\} \sin\left(\frac{n\pi x}{L}\right),$$

where $\{a_n\}$, $\{b_n\}$ are constants.

(15 marks)

(ii) Find $\{a_n\}$ and $\{b_n\}$ for the case when

$$y(x,0) = A\left[\sin\left(\frac{\pi x}{L}\right) + \sin\left(\frac{3\pi x}{L}\right)\right]; \qquad y_t(x,0) = 0,$$

where A is constant.

(10 marks)

3 (*A model of a stethoscope.*) Sound waves propagate in the positive Oz direction inside the circular cylinder $r = a$ (where $r^2 = x^2 + y^2$ in standard notation). The velocity potential ϕ satisfies

$$c^2 \left\{ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} \right\} = \frac{\partial^2 \phi}{\partial t^2}, \tag{2}$$

where the constant c is the speed of sound.

(i) State how c depends on pressure (p) and density (ρ). Determine the value of c for the case when this is

$$\left(\frac{p}{p_0}\right) = \left(\frac{\rho}{\rho_0}\right)^{\gamma},$$

where $\gamma = 1.4$, and p_0 and ρ_0 are the ambient pressure and density with $p_0 \approx 1.013 \times 10^5\ \mathrm{N\,m^{-2}}$, $\rho_0 \approx 1.293\ \mathrm{kg\,m^{-3}}$.

(6 marks)

(ii) Seek solutions of (2) of the form

$$\phi = g(r)\exp\{i(kz - \omega t)\},$$

where k and ω are real positive constants. Show that

$$g''(r) + \frac{1}{r}g'(r) + m^2 g(r) = 0 \tag{3}$$

where m^2 is a constant, depending on ω, c and k. (You may assume that $m^2 > 0$.)

(7 marks)

(iii) It is given that ϕ is bounded at $r = 0$, that $\dfrac{\partial \phi}{\partial r} = 0$ at $r = a$, and that the only solution of (3) that is bounded at $r = 0$ must be a multiple of $J_0(mr)$, where $J_0(\xi)$ is the Bessel function of order zero. Show that $m = m_n$ $(n = 1, 2, \ldots)$, where $m_n = \beta_n/a$ and β_n is the nth non-zero root of $J_0'(\xi) = 0$. Given that the β_n are discrete, that $\beta_1 < \beta_2 < \ldots$, and that $\beta_n \to \infty$ as $n \to \infty$, deduce that, for fixed ω, there are a finite number of positive values of k.

(12 marks)

4 The equilibrium position of the free surface of a liquid is $z = 0$, where z is measured vertically upwards. A short surface wave, causing the displacement of this surface to be $\eta(x, t)$, where x is measured along the undisturbed surface, is affected by a phenomenon known as *surface tension.* This results in the pressure at the free surface not being continuous. You are given, as a consequence, a linear dynamic boundary condition

$$\frac{\partial \phi}{\partial t} + g\eta = \frac{T}{\rho}\frac{\partial^2 \eta}{\partial x^2} \quad \text{at} \quad z = 0,$$

where T is the magnitude of the surface tension, and, (a) the velocity potential $\phi = \phi(x, z, t)$ satisfies $\phi_{xx} + \phi_{zz} = 0$. You are also given that (b) $\phi_z = \eta_t$ at $z = 0$.

(i) Show that for a progressive surface wave on water of depth h with $\eta = \eta_0 \sin(kx - \omega t)$, $\phi = f(z)\cos(kx - \omega t)$, and, (c) $\phi_z = 0$ at $z = -h$ the dispersion relation is

$$\omega^2 = gk\left(1 + \frac{Tk^2}{\rho g}\right)\tanh kh.$$

(10 marks)

(ii) Next, suppose kh is large (deep water). Find the dispersion relation $\omega(k)$ in this approximation.

(3 marks)

(iii) Show that the phase velocity c and the group velocity c_g satisfy

$$c = \left(\frac{gT}{\rho}\right)^{\frac{1}{4}}\left(\frac{1+p^2}{p}\right)^{\frac{1}{2}}, \qquad c_g = \frac{1}{2}c\left(\frac{1+3p^2}{1+p^2}\right) \quad \text{with} \quad p^2 = \frac{Tk^2}{\rho g}.$$

Deduce that c has a minimum c_m when $p = 1$. Show that $c_g > c$ for $p > 1$ and that $c_g < c$ for $p < 1$. Sketch the graph of c against p.

(12 marks)

5 (i) In a model of traffic flow in the direction of Ox, the density of traffic at time t is $\rho(x,t)$, the speed of traffic of density ρ is $v = v(\rho)$, the flowrate $q(\rho) = \rho v(\rho)$, and $c(\rho) = q'(\rho)$. *Given* that $\rho_t + c\rho_x = 0$, show that $c_t + c\, c_x = 0$. If $\rho(x,0) = f(x)$, deduce that in regions where $c(x,t)$ is continuously differentiable:

$$c = c\{f(\xi)\} = F(\xi) \text{ on straight lines } x = \xi + F(\xi)t.$$

(12 marks)

(ii) Using the method of characteristics solve the equation

$$yz_x + xz_y = xy,$$

given that $z = e^{-y^2}$ on $x = 0$ for $y \geq 0$ and that $z = e^{-x^2}$ on $y = 0$ for $x \geq 0$.

(13 marks)

End of Question Paper