The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2019–2020**

Further Foundation Mathematics **Submission by: 13.00 BST, 5th June 2020**

This is an open book exam.

*Answer **all** questions. Total marks 80.*
The submission deadline is 13.00 (BST), twenty-four hours after it is released. Late submission will not be considered without extenuating circumstances. It is expected that you will be able to complete this exam in approximately two hours and it is recommended that you submit the work within five hours. You will not be penalised for taking longer, however.

Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working. You will not get full marks if you simply write down output from a computer package.

By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged and that no unfair means have been used.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 (i) Express the following in the form $z = a + ib$ (where a and b are real numbers) and write down the *complex conjugate*, *modulus* and *principal argument* of each of the resulting complex numbers.

Note: for the principal argument, if it is a well known multiple of π, leave the answer in the form "(number) $\cdot \pi$". Otherwise, the answer should be given in radians with an accuracy of 4 significant digits. E.g., if you get $Arg(z) = \pi/2$ there is no need to use a calculator and express this with 4 significant digits.

(a) $z_1 = (3 + 4i) + (2 + 2i)$,

(b) $z_2 = (1 + 3i)(3 - i)$,

(c) $z_3 = \dfrac{2}{i}$,

(d) $z_4 = \dfrac{(1+i)^2}{5(1-i)}$.

(8 marks)

(ii) For the following question, $z \in \mathbb{C}$ (i.e., a complex number), while $x, y \in \mathbb{R}$ (i.e., real numbers).

(a) What is the locus described by each of the equations below? Justify your answers, and give their expression in terms of x, y, the real and imaginary parts of z.

(α) $z = 4i$;

(β) $|z - 3 - 2i| = 2$;

(γ) $|z - 2i| = |z - 2|$;

(δ) $x^2 + 6x + y^2 - 4y + 9 = 0$; ***(8 marks)***

(b) Show that the $x-$axis is a tangent line to equation (δ), and find the point of tangency. ***(4 marks)***

2 In this question, let $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$ and $\mathbf{w} = 2\mathbf{j} - 2\mathbf{k}$.

(i) Find $\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{w}$, $\mathbf{v} \cdot \mathbf{w}$, $\mathbf{u} \times \mathbf{v}$ and $\mathbf{w} \times \mathbf{w}$. ***(5 marks)***

(ii) (a) What is the angle between $\mathbf{u}$ and $\mathbf{w}$? Give your answer in radians to four significant figures. ***(4 marks)***

(b) Find a value of λ such that the vector $\mathbf{u} + \lambda\mathbf{v}$ is perpendicular to $\mathbf{w}$. ***(3 marks)***

(c) Using the cross product, find the value of μ such that $\mathbf{v} + \mu(\mathbf{2u} + \mathbf{3v})$ is parallel to $\mathbf{v}$. Any other method will receive partial marks. ***(4 marks)***

(iii) Where does the line with equation $\mathbf{x} = \mathbf{u} + \nu\mathbf{v}$ meet the plane with equation $\mathbf{x} = \mathbf{i} + \sigma\mathbf{j} + \tau\mathbf{k}$? ***(4 marks)***

3 (i) Consider the function $f(x) = |2x - 4| + 2|x|$.

(a) Write $f(x)$ as a piecewise-defined function. ***(2 marks)***

(b) By considering the limit definition of continuity, show that this function is continuous at both $x = 0$, and $x = 2$. You need to fully justify your answer. ***(2 marks)***

(ii) (a) Write down the binomial expansion for $\dfrac{1}{\sqrt{(1+4x)}}$ up to (and including) the x^4 term. Explicitly show how you find the coefficients in the expansion. ***(6 marks)***

(b) Find the first few terms in the Maclaurin series for $\cos(x)$ up to (and including) the x^4 term. You need to explicitly show your calculations.

(6 marks)

(c) Find the first few terms in the Maclaurin series for $\dfrac{\cos(x)}{\sqrt{1+4x}}$ up to (and including) the x^4 term.
Hint: use the previous questions. ***(4 marks)***

4 (i) For the following differential equations find their order, say whether they are linear or nonlinear, and (if they're linear) whether they have constant or variable coefficients.

(a) $x\dfrac{d^2y}{dx^2} - \dfrac{7}{2}\left(\dfrac{dy}{dx}\right)^3 + y = 0.$

(b) $4\dfrac{d^4y}{dx^4} + 3\dfrac{d^3y}{dx^3} = 2\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx}$

(c) $\cos(y)\dfrac{dy}{dx} - 2\dfrac{d^2y}{dx^2} + 2y = 0.$

(3 marks)

(ii) (a) Find the general solution to the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$$

(6 marks)

(b) Hence find a solution to the above equation which has $y = 3$ when $x = 0$ and $y = e^2 + 2e$ when $x = 1$. *(4 marks)*

(iii) (a) Find, by the method of separation of variables or otherwise, the solution to the differential equation

$$\frac{dy}{dx} = xy^2,$$

that also satisfies the condition $y(0) = -1$.

(7 marks)

End of Question Paper