The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2019–2020**

Mathematics Core II **24 hours**

This is an open book exam.

Section A consists of longer questions for which you should try to write answers which are clear, concise, coherent, complete and correct.

Section B consists of some multiple choice questions, covering some skills which are not addressed in Section A.

Solutions should be submitted via the Blackboard MAS111 page.

Your tutor will provide some feedback; comments will include something on the way you are writing mathematics.

The submission deadline is 10 am (BST), twenty-four hours after it is released. Late submission will not be considered without extenuating circumstances. It is expected that you will be able to complete this exam in approximately one and a half hours and it is recommended that you submit the work within four and a half hours. You will not be penalised for taking longer, however.

Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working. You will not get full marks if you simply write down output from a computer package.

By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged and that no unfair means have been used.

Section A

Remember to write your answers as clearly as possible. Your marker will comment on this too.

A1 Use row operations to find the value of k for which the following system of equations has solutions:

$$\begin{aligned} x + y + z &= 4 \\ -2x + y + 4z &= 16 \\ 7x + y - 5z &= 4k. \end{aligned}$$

For this value of k, what does the set of solutions look like geometrically?

Write a short paragraph about linear independence of equations and how it relates to solution sets, illustrating it by referring to this set of equations.

A2 Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

(i) Work out A^2 and A^{-1}.

(ii) What is $\det(A^7)$? Justify your answer briefly.

(iii) Compute the eigenvalues and eigenvectors for A.

(iv) Write down the eigenvalues of $B = A^2 - I + 3A^{-1}$.

(v) Write down a matrix M such that $M^{-1}BM$ is diagonal, justifying your answer briefly.

(vi) Explain how the product $M^{-1}BM$ acts on the standard basis of $\mathbb{R}^2$.

A3 Recall that if A is a square matrix, then we define $e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$. Using the Taylor series about 0 for $\sinh x$ and $\cosh x$, show that

$$\exp\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} = \begin{pmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{pmatrix}.$$

A4 Let $f(x, y) = e^y(\sin x + \cos x)$. Compute the Taylor series around $(0, 0)$ up to the degree 2 terms, and derive an approximation to $f(0.1, -0.2)$.

Using this technique, what could you do to get a better approximation?

A5 Let $f(x,y) = (2x+1)y^2 + 2xy$. Show that this function has two stationary points, both of which are saddle points.

If we had a function $f(x,y,z)$ of three variables, what would you expect the conditions to be for (x_0, y_0, z_0) to be a maximum point?

A6 Evaluate $\iint_T e^{(x-y)/(x+y)} dA$, where T denotes the triangle with vertices $(0,0)$, $(2,0)$ and $(0,2)$, using the substitution $u = x - y$, $v = x + y$.

Section B

Each question or incomplete statement in this section is followed by four possible options of which exactly one is correct. Write down what you believe to be the correct answer.

B1 Three linear equations in the five variables x, y, z, t and u are solved using complete elimination, ending with the matrix $\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$. Which variables should you choose for parameters?

A. y, t and u **B.** x, y and z **C.** x and z **D.** t and u

B2 A is a 2×3-matrix, B is a 3×1-matrix and C is a 2×3-matrix. Which of the following products makes sense?

A. BA^T **B.** CB^T **C.** $B^T A$ **D.** $C^T A$

B3 What is the determinant of $\begin{pmatrix} a & b & c \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$?

A. $2a + 6b + 3c$ **B.** $-7a + 5b - c$ **C.** $2a - 6b + 3c$ **D.** $3a - b + 2c$

B4 The numbers a, b and c are such that $\begin{vmatrix} a & b & c \\ 1 & 2 & 3 \\ -1 & 2 & 0 \end{vmatrix} = -3$. Let $d_1 = \begin{vmatrix} 1 & 2 & 3 \\ a & b & c \\ -1 & 2 & 0 \end{vmatrix}$, $d_2 = \begin{vmatrix} a & -b & c \\ 1 & -2 & 3 \\ -1 & -2 & 0 \end{vmatrix}$, $d_3 = \begin{vmatrix} -a & -b & -c \\ -1 & -2 & -3 \\ 1 & -2 & 0 \end{vmatrix}$ and $d_4 = \begin{vmatrix} a & b & c \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix}$. Which of d_1, d_2, d_3 and d_4 are positive?

A. d_1 and d_3 **B.** d_2 and d_4 **C.** d_1, d_2 and d_4 **D.** d_1, d_2 and d_3

B5 Write down the characteristic polynomial of $\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$.

A. $\lambda^2 - 7\lambda + 10$ **B.** $\lambda^2 - \lambda - 3$ **C.** $\lambda^2 - 3\lambda - 10$ **D.** $\lambda^2 - 3\lambda + 10$

B6 What is the sum of the eigenvalues of $\begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}$?

A. 1 **B.** -7 **C.** 7 **D.** 2

B7 What is the tangent plane to $z = x + 2\sin(x + y)$ at $(0, 0, 0)$?

A. $2x+2y+z=4$ **B.** $x+y+z=0$ **C.** $z=3x+2y$ **D.** $2x+3y-z=0$

B8 A real symmetric 3×3-matrix has two eigenvectors $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$. What is the other?

A. $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ **B.** $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ **C.** $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ **D.** $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

B9 The parallelogram P has vertices at $(0, 0)$, $(1, 1)$, $(0, 2)$ and $(1, 3)$. We can compute its area with

A. $\int_0^1 \int_x^{x+2} 1\, dy\, dx$ **B.** $\int_0^1 \int_0^3 1\, dx\, dy$ **C.** $\int_{y-2}^{y} \int_0^1 1\, dy\, dx$ **D.** $\int_0^3 \int_x^{x+2} 1\, dx\, dy$

B10 The integral $\int_{-1}^{1}\int_{y}^{1} y\,dx\,dy$ can also be computed with

A. $\int_{y}^{1}\int_{-1}^{1} y\,dy\,dx$ **B.** $\int_{-1}^{1}\int_{x}^{1} y\,dy\,dx$ **C.** $\int_{-1}^{y}\int_{-1}^{x} y\,dy\,dx$ **D.** $\int_{-1}^{1}\int_{-1}^{x} y\,dy\,dx$

End of Question Paper