The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2019–2020**

Fluid Mechanics I **1 hour**

This is an open book exam.
Answer ***both*** *questions. The marks awarded to each section of question are shown in italics. The total mark for the paper is 50.*
The submission deadline is 10 am (BST), twenty-four hours after it is released. Late submission will not be considered without extenuating circumstances. It is expected that you will be able to complete this exam in approximately one hour and it is recommended that you submit the work within four hours. You will not be penalised for taking longer, however.
Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working. You will not get full marks if you simply write down output from a computer package.
By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged and that no unfair means have been used.

1 An incompressible fluid of constant density ρ flows steadily between slightly porous horizontal solid boundaries at $y = 0$ and $y = a\,(> 0)$. Fluid percolates in through the wall at $y = 0$ and out through the wall at $y = a$ at the same constant positive speed v, while the main flow is along the channel at speed $U(y)$. The total fluid velocity can be taken to be

$$\mathbf{u} = U(y)\mathbf{i} + v\mathbf{j},$$

with boundary conditions

$$U = 0 \qquad \text{at } y = 0 \text{ and at } y = a.$$

Body forces can be ignored.

(a) Deduce that

$$\frac{\partial p}{\partial x} = -G$$

for some constant G. ***(7 marks)***

If $G > 0$, what sign would you expect U to have? ***(1 mark)***

(b) Show that

$$U(y) = \frac{Ga}{\rho v}\left(\frac{y}{a} - \frac{1 - e^{vy/\nu}}{1 - e^{va/\nu}}\right),$$

where ν is the kinematic viscosity. ***(13 marks)***

(c) If $\dfrac{va}{\nu} \gg 1$, sketch $\dfrac{\rho v U}{Ga}$ against $\dfrac{y}{a}$, giving an argument for why it looks like that. ***(4 marks)***

2 An incompressible fluid of constant density ρ is contained between two circular cylinders, both centred on the z-axis. The inner cylinder has radius a_1 and rotates with constant angular velocity Ω. The outer cylinder has radius $a_2\,(> a_1)$ and is stationary. The flow is steady and body forces can be ignored. In cylindrical polar coordinates (r, θ, z) the velocity is

$$\mathbf{u} = u(r)\hat{\boldsymbol{\theta}}.$$

(a) Use the identities

$$\mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{u})$$

$$\nabla^2 \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u})$$

with

$$\nabla \times \mathbf{u} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ u_r & r u_\theta & u_z \end{vmatrix},$$

where $\mathbf{u} = u_r\hat{\mathbf{r}} + u_\theta\hat{\boldsymbol{\theta}} + u_z\hat{\mathbf{z}}$, to show that

$$\begin{cases} \dfrac{\partial p}{\partial r} = \dfrac{\rho u^2}{r} \\ \dfrac{\partial p}{\partial \theta} = \mu r \dfrac{d}{dr}\left\{\dfrac{1}{r}\dfrac{d}{dr}(ru)\right\} \\ \dfrac{\partial p}{\partial z} = 0 \end{cases}$$

(10 marks)

(b) Use the result of part (a) to deduce that

$$\frac{\partial p}{\partial \theta} = 0.$$

(3 marks)

Hence show that

$$u = \frac{\Omega a_1^2}{a_2^2 - a_1^2}\left(\frac{a_2^2 - r^2}{r}\right).$$

(7 marks)

(c) If $u = \frac{1}{3}\Omega a_1$ when $r = 2a_1$, with $2a_1 < a_2$, find a_2/a_1. ***(5 marks)***

End of Question Paper