Data provided: Formulae Sheet

MAS340

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS

Spring Semester 2019–2020

Mathematics (Computational Methods)

Nominal 1 hour in a 24 hour period

This is an open book exam.
*Answer **all** questions.*
The submission deadline is 10 am (BST), twenty-four hours after it is released. Late submission will not be considered without extenuating circumstances. It is expected that you will be able to complete this exam in approximately one hour and it is recommended that you submit the work within four hours. You will not be penalised for taking longer, however.
Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working. You will not get full marks if you simply write down output from a computer package.
By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged and that no unfair means have been used.
The paper will be marked out of 50 and the allocation of marks is shown in brackets.

Registration number from U-Card (9 digits)
to be completed by student

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1 (i) (a) The function $u(x,t)$ satisfies the heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

together with the conditions

$$u(x,0) = 18 - 24\left|x - \frac{1}{4}\right|, \quad 0 < x < 1,$$

$$u(0,t) = 12,$$

$$u(1,t) = 0.$$

You are **given** that the Crank-Nicolson scheme for this heat conduction equation has the form

$$-ru_{i-1,j+1} + (2+2r)u_{i,j+1} - ru_{i+1,j+1} = ru_{i-1,j} + (2-2r)u_{i,j} + ru_{i+1,j},$$

where, in the usual notation, $r = \dfrac{k}{h^2}$.

For the case where $h = 0.25$ and $k = 0.0625$, set up a table showing the values of u at the grid points for $t = 0$.

Hence calculate, to TWO decimal places, the values of u at the gridpoints for $t = 0.0625$.

You are **given** that

$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}^{-1} = \frac{1}{56}\begin{bmatrix} 15 & 4 & 1 \\ 4 & 16 & 4 \\ 1 & 4 & 15 \end{bmatrix}$$

(10 marks)

 Question 1 continued on next page

1 (continued)

(ii) (a) State briefly the key benefit of using cubic spline interpolation rather than some other interpolating polynomials, such as Lagrange polynomials or Newton polynomials. **(1 mark)**

Explain briefly the term 'natural conditions' in relation to cubic splines. **(2 marks)**

(b) Let $a = x_0 < x_1 < \cdots < x_N = b$, and let $f(x_i) = f_i$ for function f which is continuous on $[a, b]$.
Let S be the cubic spline interpolant to f at the equally spaced points, $x_i,\ i = 0, 1, \cdots, N$.
The cubic interpolant defined in the interval $[x_i, x_{i+1}]$ has the form

$$s_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i.$$

The coefficients of the interpolant polynomial are defined as:

$$a_i = \frac{\sigma_{i+1} - \sigma_i}{6h},$$

$$b_i = \frac{\sigma_i}{2},$$

$$c_i = \frac{f_{i+1} - f_i}{h} - \frac{h}{6}(2\sigma_i + \sigma_{i+1}),$$

$$d_i = f_i,$$

where $h = x_{i+1} - x_i$, f_i is the value of the function at the i-th position, $s''(x_i) = \sigma_i$, where $'$ denotes differentiation with respect to x, and $0 \le i \le N$.

Given that

$$\sigma_{i-1} + 4\sigma_i + \sigma_{i+1} = \frac{6}{h^2}\left(f_{i-1} - 2f_i + f_{i+1}\right),$$

determine the natural cubic spline approximation between the following data points

x	-1.0	0.0	1.0	2.0
$f(x)$	1.937	1.0	1.349	-0.995

Work correct to FOUR decimal places. **(9 marks)**

(c) Use the cubic spline interpolant to estimate $f(1.5)$ and $f'(1.5)$. **(3 marks)**

2 (i) (a) Find and classify the stationary points of the function

$$f(x, y) = 4x^2 + 2xy + 2.5y^2.$$

(6 marks)

(b) Calculate an approximation to the minimum of $f(x, y)$ using one iteration of Newton's method starting from the point $(10, 10)$.
(7 marks)

(ii) A company plans to build **seven** hospitals on selected sites during the next **four** years. The construction of each hospital is completed within one year, where each period of one year starts on 1 January and ends on 31 December. Let

$$x_{ij} = \begin{cases} 1 & \text{if hospital } i \text{ is constructed in year } j \\ 0 & \text{if hospital } i \text{ is not constructed in year } j \end{cases}$$

and c_{ij} = cost of building factory i in year j.

Set up the integer programming problem to minimise the total costs over the four year period provided all the factories are built. **(2 marks)**

For each of the **separate** cases below, list the additional constraints required.

(a) Hospital 2 cannot be built before hospital 6, but they can be built in the same year. **(2 marks)**

(b) Hospitals 1, 3, 7 must be built by the end of year 3. **(1 mark)**

(c) If hospitals 2 and 3 are built in a given year, then no other hospitals can be built in that year. **(4 marks)**

(d) In the first three years either hospitals 1 and 2, or, hospitals 5 and 6 must be built. **(3 marks)**

End of Question Paper

Formulae Sheet

Notation:

$$U(x_i,\, t_j) \equiv U_{i\,j}$$

Forward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{i,j+1} - U_{ij}}{\Delta t}$$

Backward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{ij} - U_{i,j-1}}{\Delta t}$$

Central difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x} \approx \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}$$

Central difference formula for $\partial^2 U/\partial x^2$:

$$\frac{\partial^2 U}{\partial x^2} \approx \frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{\Delta x^2}$$