The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2019–2020**

Graph Theory

This is an open book exam.

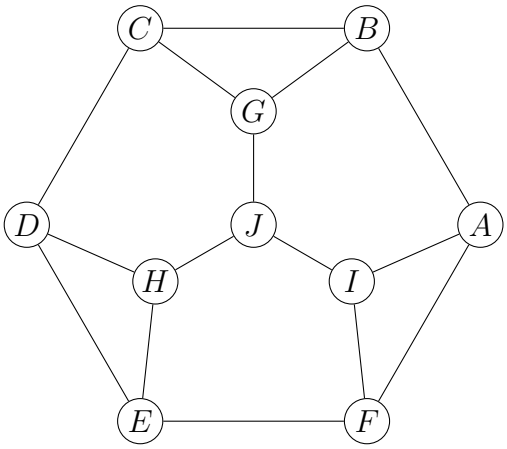
*Answer **all** questions.*

The submission deadline is 10 am (BST), twenty-four hours after it is released. Late submission will not be considered without extenuating circumstances. It is expected that you will be able to complete this exam in approximately one and a half hours and it is recommended that you submit the work within four hours. You will not be penalised for taking longer, however.

Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working. You will not get full marks if you simply write down output from a computer package.

By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged and that no unfair means have been used.

1 Question one uses the graph Γ_1 shown below.

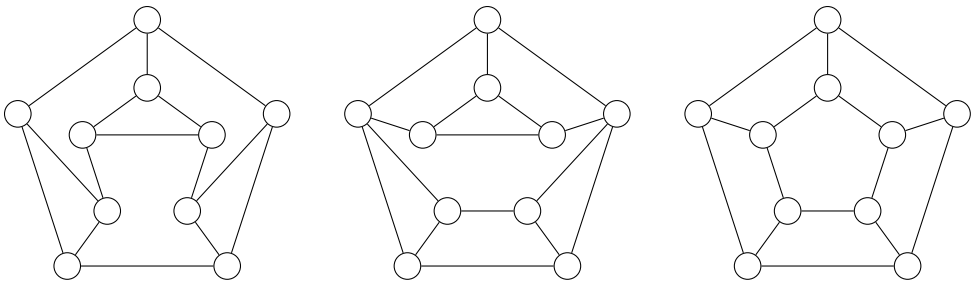


(i) Find a Hamiltonian cycle in Γ_1, and explain why Γ_1 does not have an Eulerian path. **(6 marks)**

(ii) What is the maximum number of edges in Γ_1 that can be used in a walk that does not repeat edges? Prove your answer is correct. **(4 marks)**

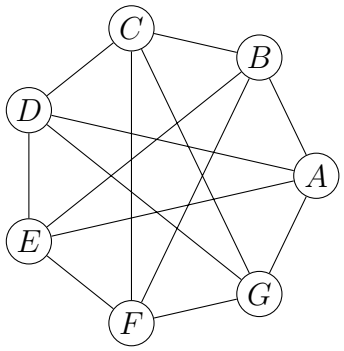
(iii) Give an isomorphism between Γ_1 above and one of the graphs below, and prove that Γ_1 is not isomorphic to either of the other two graphs. **(7 marks)**

G_1 G_2 G_3



(iv) Let X be an unknown atom. Use the handshaking lemma to prove that CX_3H_7 is a tree if and only if X has valency 3. Then, given that X has valency 3 and that CX_3H_7 is a tree, draw all of the isomers of CX_3H_7. **(8 marks)**

2 Question two uses the graph Γ_2 shown below. It also makes use of two different Hamiltonian cycles in Γ_2: the edges around the outside form the *circle* $ABCDEFGA$, while the edges on the inside form the *star* $ADGCFBEA$.

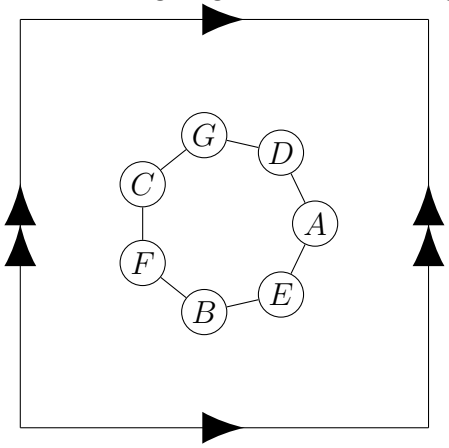


(i) Prove that Γ_2 isn't planar using Kuratowski's Theorem. Further prove that if e is any edge in the *star* Hamiltonian cycle $ADGCFBEA$, then $\Gamma_2 \setminus e$ **is not** planar. (Recall that $\Gamma_2 \setminus e$ denotes Γ_2 with the edge e deleted).

(5 marks)

(ii) Give another proof that Γ_2 isn't planar, this time using the planarity algorithm for Hamiltonian graphs **using the *star*** $ADGCFBEA$ **as the Hamiltonian cycle**. Further prove that if e is any edge in the *circle* Hamiltonian cycle $ABCDEFGA$, then $\Gamma_2 \setminus e$ **is** planar. **(6 marks)**

(iii) All of the vertices and half of the edges of Γ_2 are drawn on the torus as shown below. Add the remaining edges so that no edges cross.



(3 marks)

(iv) Find the chromatic number $\chi(\Gamma_2)$ *and* the chromatic index $\chi'(\Gamma_2)$ of Γ_2 Justify your answers. **(8 marks)**

(v) What is the coefficient of k^6 in the chromatic polynomial $P_{\Gamma_2}(k)$?

(3 marks)

End of Question Paper