The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2019–2020**

Groups and Symmetry

This is an open book exam.
*Answer **all** questions.*
The submission deadline is 10 am (BST), twenty four hours after it is released. Late submission will not be considered without extenuating circumstances. It is expected that you will be able to complete this exam in approximately one and a half hour and it is recommended that you submit the work within four hours. You will not be penalised for taking longer, however.
Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working. You will not get full marks if you simply write down output from a computer package.
By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged and that no unfair means have been used.

1 (i) Consider the group of symmetries D_4 of the polygon X_4 with vertices

$$(1, 0), (0, 1), (-1, 0), (0, -1).$$

(a) Show that the symmetric group S_4 has a subgroup H isomorphic to D_4. ***(4 marks)***

(b) Show that all Sylow-2 subgroups of S_4 are isomorphic to D_4. ***(2 marks)***

(c) Let n_2 be the number of Sylow-2 subgroups of S_4. Given that there are precisely 16 elements in S_4 whose orders divide 8, show that $n_2 > 1$ and explain why this means that H is not a normal subgroup. ***(4 marks)***

(d) Show that S_4 has four Sylow-3 subgroups. ***(2 marks)***

 Question 1 continued on next page

1 (continued)

(ii) Let G be group whose order $|G|$ is $p^\alpha q_1^{\beta_1} q_2^{\beta_2} \ldots q_k^{\beta_k}$ where $p < q_1 < q_2 < \cdots < q_k$ are prime, and $\alpha, \beta_1, \beta_2, \ldots, \beta_k > 0$ are integers. Let H be a subgroup of G whose index $[G : H]$ is p.

(a) Consider the action of G on the cosets of H by left multiplication. By applying Proposition 2.9 or otherwise, show that this action gives a homomorphism ϕ from G to the symmetric group S_p. ***(2 marks)***

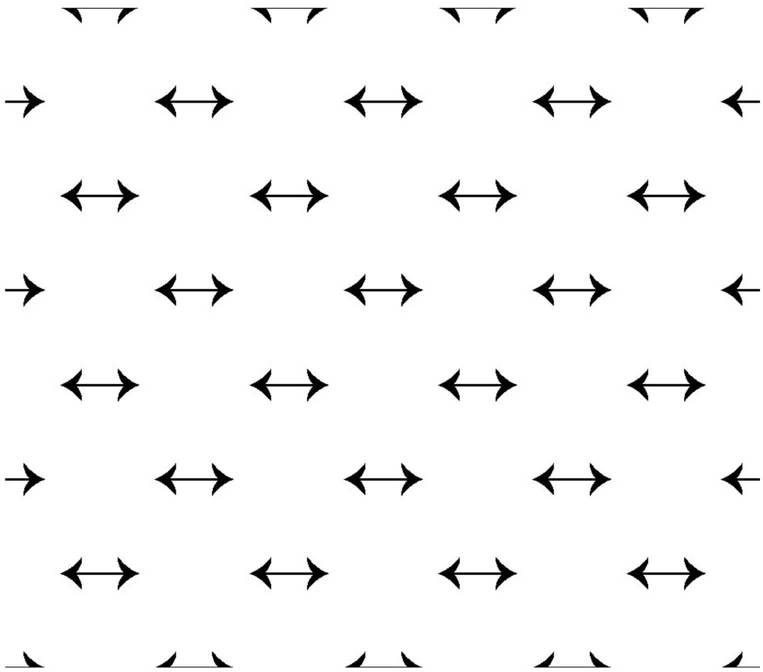
(b) Let $K = \ker \phi$. Show that $K \subseteq H$ and hence that p divides $[G : K]$, and that no smaller prime divides $[G : K]$. ***(3 marks)***

(c) Use the First Isomorphism Theorem to show that $[G : K] = |G/K|$ divides $|S_p| = p!$. ***(2 marks)***

(d) Show that the only prime divisor of $[G : K]$ which divides $p!$ is p and deduce that $[G : K]$ is 1 or p. ***(3 marks)***

(e) Combine (b) and (d) to conclude that $[G : K] = p$, hence $[G : K] = [G : H]$ and $H = K$. Deduce that H is a normal subgroup of G. ***(3 marks)***

2 Let G be the isometry group of the infinite wallpaper pattern, a portion of which is illustrated below. (A copy of the diagram on is provided at the end of the exam paper; if you wish, you may write on it and hand it in with your answer.)



(i) Describe geometrically *all* the translations, rotations and reflections in G. State clearly the vectors of the translations, the centres and angles of the rotations, and the lines of the reflections. Specify one more element of G that is not a translation, rotation or reflection. Justify your answers. ***(9 marks)***

(ii) Find a list of four isometries that generate G. Justify your answer. ***(11 marks)***

(iii) Show that the point group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. ***(5 marks)***

End of Question Paper

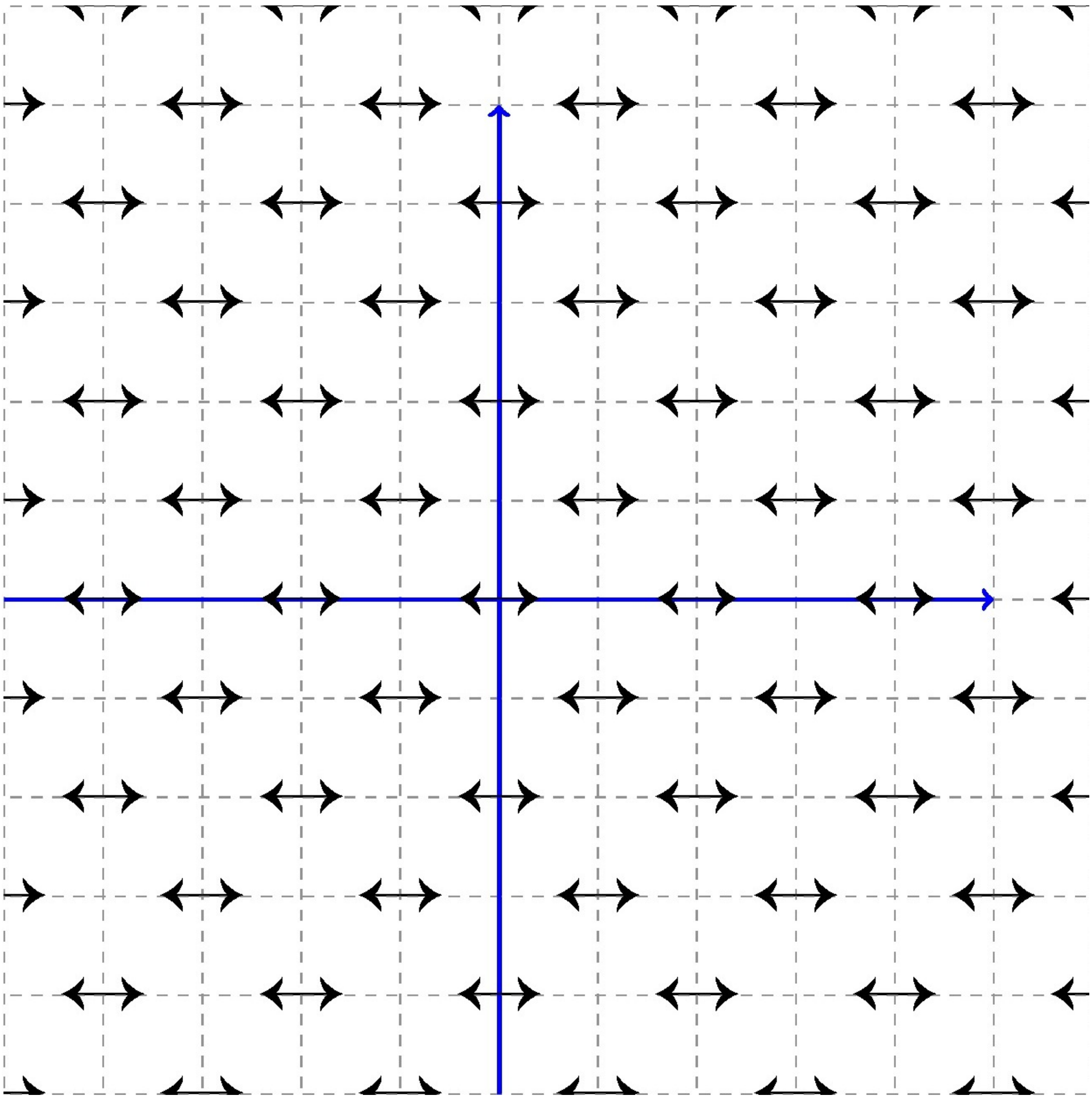


Figure 1: Wallpaper for Question 2, with axes and grid