The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2019–2020**

Probability with Measure **1 hour (nominal)**

Candidates should attempt ***ALL*** *questions.*
The maximum marks for the various parts of the questions are indicated.
The paper will be marked out of 35.

This is an open book exam.

The submission deadline is 10 am (BST), twenty-four hours after it is released. Late submission will not be considered without extenuating circumstances. It is expected that you will be able to complete this exam in approximately 1 hour and it is recommended that you submit the work within four hours. You will not be penalised for taking longer, however.

You may use a calculator, but to gain full marks you will need to show your working. You will not get full marks if you simply write down output from a computer package.

By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools, other than material provided for this module, is cited and acknowledged and that no unfair means have been used.

1 Let λ denote Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Consider the set

$$C = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \left[k, k + \tfrac{1}{n^2+k}\right]$$

and define $C_n = \bigcup_{k=1}^{\infty} \left[k, k + \frac{1}{n^2+k}\right]$.

(a) (i) Show that C is a Borel set.

You may use standard facts concerning the Borel σ-field on $\mathbb{R}$, providing they are clearly indicated.

(ii) Show that (C_n) is a decreasing sequence of sets.

(3 marks)

(b) Consider the following twelve statements.

A. For $n, k, k' \in \mathbb{N}$ we have $\left[k, k + \frac{1}{n^2+k}\right] \cap \left[k', k' + \frac{1}{n^2+k'}\right] = \emptyset$ if $k \neq k'$.

B. For each $n, k \in \mathbb{N}$ we have $\bigcap_{n=1}^{\infty} \left[k, k + \frac{1}{n^2+k}\right] = \{k\}$.

C. Recall that $\lambda([a, b]) = b - a$, for $-\infty < a \leq b < \infty$.

D. Recall that, if $(A_n) \subseteq \mathcal{B}(\mathbb{R})$ is a decreasing sequence of sets, then $\lambda(A) = \lim_{n\to\infty} \lambda(A_n)$.

E. Recall that, if (A_n) are disjoint Borel sets, then $\lambda(A) = \sum_n \lambda(A_n)$.

F. Recall that $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$.

G. By part (ii) of (a) the Lebesgue measure of C is $\lim\limits_{n\to\infty} \lambda(C_n)$.

H. Hence, for each $n \in \mathbb{N}$ we have $\lambda(C_n) = \sum_{k=1}^{\infty} \frac{1}{n^2+k}$.

I. Hence C is equal to $\mathbb{N}$.

J. Hence C is equal to $[0, \infty)$.

K. Hence, $\lambda(C) = \sum_{k=1}^{\infty} 0 = 0$.

L. Hence, $\lambda(C) = \infty$

Six of the statements **A**-**L** can be used, when arranged into order, to give a proof of the correct value of $\lambda(C)$. The other six statements are not required.

List which six statements are required. ***(5 marks)***

2 (a) Let $g : \mathbb{R} \to \mathbb{R}$ be given by

$$g(x) = \begin{cases} -1 & \text{if } x \in [-1, 1) \\ 2 & \text{if } x \in [2, 3) \\ -4 & \text{if } x \in [3, 4) \\ 0 & \text{otherwise} \end{cases}$$

(i) Write g and e^g explicitly as simple functions, where e^g is defined pointwise: $e^g(x) = e^{g(x)}$.

(ii) Verify that

$$\exp\left(\int_{\mathbb{R}} g(x)\, dx\right) \leq \int_{\mathbb{R}} e^{g(x)}\, dx$$

by computing each of the integrals explicitly.

(4 marks)

(b) (i) For $i = 1, \ldots, n$ let $c_i \in \mathbb{R}$ and let $\alpha_i \in [0, 1]$ with $\sum_{i=1}^{n} \alpha_i = 1$. Show that

$$1 \leq \sum_{i=1}^{n} \alpha_i \exp\left(c_i - \sum_{j=1}^{n} \alpha_j c_j\right).$$

Hint: Use that $e^x \geq 1 + x$ for all $x \in \mathbb{R}$.

(ii) Let m be a probability measure on the measurable space (S, Σ). Show that

$$\exp\left(\int_S f\, dm\right) \leq \int_S e^f\, dm \qquad (\star)$$

for all non-negative measurable functions $f : S \to \mathbb{R}$.

(iii) Give an example of a finite measure space (S, Σ, m) and a function f such that both sides of $(\star)$ are equal. ***(9 marks)***

3 (a) (i) Show that the function $f : [0, \infty) \to \mathbb{R}$ given by $f(x) = e^{-x}$ is integrable.

(ii) For $n \in \mathbb{N}$ let $f_n : [0, \infty) \to \mathbb{R}$ be given by

$$f_n(x) = e^{-x} e^{\sin(x/n)}.$$

Show that, as $n \to \infty$,

$$\int_0^\infty f_n(x)\, dx \to 1.$$

(5 marks)

(b) Both of the following claims are *false*. In each case, find a counterexample.

(i) If $f_n : \mathbb{R} \to \mathbb{R}$ are indicator functions such that $f_n(x) \to 0$ for all $x \in \mathbb{R}$, then

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\, dx = 0.$$

(ii) If $f : \mathbb{R} \to [0, \infty)$ satisfies $0 < \int_{\mathbb{R}} f(x)\, dx < \infty$ then

$$\limsup_{n\to\infty} \frac{f(x_n)}{x_n} = 0$$

for any sequence $(x_n) \subseteq [0, \infty)$ such that $x_n \to \infty$. ***(3 marks)***

(c) Let $f : [0, \infty) \to [0, \infty)$ be integrable. Prove that there exists a sequence (x_n) such that $x_n f(x_n) \to 0$ as $n \to \infty$ ***(2 marks)***

4 For each $i = 1, 2, 3$, let $(Y_n^{(i)})_{n\in\mathbb{N}}$ be a sequence of random variables with distribution

$$\mathbb{P}\left[Y_n^{(i)} = 0\right] = \frac{1}{\sqrt{n}}, \qquad \mathbb{P}\left[Y_n^{(i)} = 1\right] = 1 - \frac{1}{\sqrt{n}}.$$

We assume that $Y_n^{(i)}$ and $Y_m^{(j)}$ are independent whenever $(i, n) \neq (j, m)$.

Show that

$$\mathbb{P}\left[Y_n^{(1)} = Y_n^{(2)} = 0 \text{ infinitely often}\right] = 1$$

and

$$\mathbb{P}\left[Y_n^{(1)} = Y_n^{(2)} = Y_n^{(3)} = 0 \text{ infinitely often}\right] = 0.$$

(4 marks)

End of Question Paper