The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2019–2020**

Analytical Dynamics and Classical Field Theory **1 hour 30 minutes**

This is an open book exam.

*Answer **all** questions.*

The submission deadline is 10 am (BST), twenty-four hours after it is released. Late submission will not be considered without extenuating circumstances. It is expected that you will be able to complete this exam in approximately one and a half hours and it is recommended that you submit the work within four and a half hours. You will not be penalised for taking longer, however.

Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working. You will not get full marks if you simply write down output from a computer package.

By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged and that no unfair means have been used.

1 (i) The Lagrangian $L_1(q, \dot{q}, t)$ for a mechanical system with one degree of freedom is given by

$$L_1(q, \dot{q}, t) = \dot{q}^2 + \omega^2 q^2,$$

where ω is a constant, q is the generalized coordinate, $\dot{q}$ is the generalized velocity and a dot denotes derivative with respect to time t.

Write down Lagrange's equation for the system. ***(2 marks)***

Find the Hamiltonian of the system. ***(2 marks)***

Write down Hamilton's equations for the system. ***(2 marks)***

Find a constant of the motion for this system. ***(1 mark)***

(ii) The Lagrangian $L_2(q, \dot{q}, t)$ for a second mechanical system with one degree of freedom is given by

$$L_2(q, \dot{q}, t) = \dot{q}^2 \sin^2(\omega t) + \omega q \dot{q} \sin(2\omega t) + \omega^2 q^2,$$

where ω is a constant, q is the generalized coordinate, $\dot{q}$ is the generalized velocity and a dot denotes derivative with respect to time t.

Show that the Hamiltonian of the second mechanical system is given by

$$H_2(q, p, t) = \frac{p^2}{4\sin^2(\omega t)} - \frac{\omega q p \cos(\omega t)}{\sin(\omega t)} - \omega^2 q^2 \sin^2(\omega t). \qquad (*)$$

(8 marks)

(iii) Let $F(q, p, Q, P, t)$ be a generating function for a canonical transformation from canonical variables (q, p) to new variables (Q, P) such that $\widetilde{H}(Q, P, t)$ is the transformed Hamiltonian of the system.

If $F(q, p, Q, P, t)$ has the form

$$F = -QP + g(q, P, t)$$

for some function $g(q, P, t)$, show that g must satisfy the partial differential equations

$$\frac{\partial g}{\partial q} = p, \qquad \frac{\partial g}{\partial P} = Q, \qquad \frac{\partial g}{\partial t} = \widetilde{H} - H.$$

(3 marks)

If $g(q, P, t) = qP \sin(\omega t)$, find Q and P in terms of q and p. ***(2 marks)***

If the Hamiltonian $H(q, p, t)$ is given by (*), find $\widetilde{H}(Q, P, t)$. ***(3 marks)***

What does your result for $\widetilde{H}$ imply about the two mechanical systems considered in parts (i) and (ii)? ***(1 mark)***

Use your result to write down, as a function of p, q and t, a constant of the motion for the mechanical system considered in part (ii). ***(1 mark)***

2 (i) Let A^μ be a vector field, B_μ be a covector field, and ∇_μ be the covariant derivative which is Leibniz and metric-compatible. The covariant derivative of B_μ is

$$\nabla_\nu B_\mu = \partial_\nu B_\mu - \Gamma^\sigma{}_{\mu\nu} B_\sigma.$$

(a) Is the affine connection $\Gamma^\sigma{}_{\mu\nu}$ a tensor? Briefly justify your answer. ***(2 marks)***

(b) By considering the gradient of the scalar $A^\mu B_\mu$, or otherwise, derive that the covariant derivative of A^μ is

$$\nabla_\nu A^\mu = \partial_\nu A^\mu + \Gamma^\mu{}_{\sigma\nu} A^\sigma.$$

(4 marks)

(ii) A static spherically-symmetric spacetime has the squared line element

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

Affinely-parameterized geodesics $x^\mu(\lambda)$ are determined from the Lagrangian $L = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$, where $\dot{x}^\mu = dx^\mu/d\lambda$.

(a) Calculate the components $\Gamma^t{}_{rr}$ and $\Gamma^r{}_{tt}$ for this spacetime. ***(3 marks)***

(b) Write down the Euler-Lagrange equations. ***(1 mark)***

Show that E and h are constant along a geodesic, where $E \equiv F(r)\dot{t}$ and $h \equiv r^2 \sin^2\theta\, \dot{\phi}$. ***(3 marks)***

Derive an acceleration equation for timelike geodesics in the equatorial plane, in the form

$$\ddot{r} = -\frac{1}{2} V'(r)$$

where $V(r)$ is a function you should determine. ***(3 marks)***

(iii) A spaceship hovers at fixed coordinates $r = r_0 \equiv 6M$, $\theta = \pi/2$, $\phi = 0$ near a Schwarzschild black hole with $F(r) = 1 - 2M/r$. A beacon measuring proper time τ is released from the spaceship at $\tau = 0$ and fired toward the centre of the black hole, with initial velocity $\dot{r} = -\sqrt{1/3}$.

What is the value of τ on the beacon's clock when it crosses the event horizon? ***(4 marks)***

Upon reaching $r = 3M$, the beacon emits a pulse of light, at $T = 0$ on the spaceship's clock. At what time T does the spaceship receive this pulse? ***(5 marks)***

End of Question Paper