MAS436

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2019–2020**

MAS436: Functional Analysis **1 hours 30 minutes**

This is an open book exam.
*Answer **all** questions.*
The submission deadline is 10 am (BST), twenty-four hours after it is released. Late submission will not be considered without extenuating circumstances. It is expected that you will be able to complete this exam in approximately one and a half hour and it is recommended that you submit the work within four hours. You will not be penalised for taking longer, however.
Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working. You will not get full marks if you simply write down output from a computer package.
By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged and that no unfair means have been used.

1 (i) (a) Let V and W be normed vector spaces. Say what is meant when we describe a linear map $T\colon V \to W$ as *bounded*. ***(2 marks)***

(b) State the closed graph theorem. ***(3 marks)***

(c) Let H be a Hilbert space. Let $S, T\colon H \to H$ be linear maps where

$$\langle Sv, w\rangle = \langle v, Tw\rangle$$

for all $v, w \in H$. Prove that S and T are bounded linear maps. ***(6 marks)***

(d) Give an example of a linear map between normed vector spaces that does not have a closed graph. Justify your answer. ***(5 marks)***

(ii) (a) Let A be a unital Banach algebra. Define the spectrum of an element $x \in A$, and state the spectral mapping theorem for polynomials. ***(4 marks)***

(b) Let H be a Hilbert space, and let $P\colon H \to H$ be a *projection*, that is to say a bounded linear map such that $P^* = P = P^2$. Show that $Spectrum(P) \subseteq \{0, 1\}$. ***(5 marks)***

2 (i) State what is meant by saying a normed vector space is a Banach space.

(2 marks)

(ii) Let V be a Banach space, and let (v_n) be a sequence in V. State what is meant when we say the series

$$\sum_{n=0}^{\infty} v_n$$

converges.

By showing that the sequence of partial sums is a Cauchy sequence or otherwise, show that if the series

$$\sum_{n=0}^{\infty} \|v_n\|$$

converges in $\mathbb{R}$, then the series

$$\sum_{n=0}^{\infty} v_n$$

converges in V. ***(6 marks)***

(iii) Let H be a Hilbert space, and let $T\colon H \to H$ be a bounded linear map. Define the *adjoint* of T. Say what is meant by saying that T is

(a) self-adjoint and

(b) unitary. ***(4 marks)***

(iv) Let $T\colon H \to H$ be a bounded linear map. Show, using part (ii) or otherwise, that the series

$$\exp(T) = \sum_{n=0}^{\infty} \frac{T^n}{n!}$$

converges. ***(5 marks)***

(v) Suppose $A, B\colon H \to H$ are bounded linear maps such that $AB = BA$. Show that

$$\exp(A)\exp(B) = \exp(A + B)$$

(4 marks)

(vi) Show that if $A\colon H \to H$ is self-adjoint, then the operator $\exp(iA)\colon H \to H$ is unitary. You may use standard properties of adjoints of operators without proof.

(4 marks)

End of Question Paper