The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2019–2020

Optics and Symplectic Geometry — 1 hours and 30 minutes

This is an open book exam.
*Answer **all** questions.*
The submission deadline is 10 am (BST), twenty-four hours after it is released. Late submission will not be considered without extenuating circumstances. It is expected that you will be able to complete this exam in approximately one and a half hour and it is recommended that you submit the work within four hours. You will not be penalised for taking longer, however.
Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working. You will not get full marks if you simply write down output from a computer package.
By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged and that no unfair means have been used.

Through the paper I denotes an identity matrix and J denotes a matrix of the form $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. All matrices have real entries. The standard symplectic form Ω on $\mathbb{R}^{2n}$ is defined by $\Omega(Z, Z') = Q \cdot P' - P \cdot Q'$, where $Z = (Q, P)$ and $Z' = (Q', P')$ are elements of $\mathbb{R}^{2n}$.

1 (i) Define $F_1(x,y,z) = x^2 + y^2 + z^2 - 4$, $F_2(x,y,z) = x^2 + (y-a)^2 - 1$, and $F(x,y,z) = (F_1(x,y,z), F_2(x,y,z))$, and write $M_1 = F_1^{-1}(0)$, $M_2 = F_2^{-1}(0)$, and $M = F^{-1}(0,0) = M_1 \cap M_2$.

(a) Show that M_1 and M_2 are manifolds. ***(8 marks)***

(b) You are given that $M \neq \emptyset$ when $a \in]-3,3[$. Find all values of a such that M is a manifold. ***(15 marks)***

(ii) (a) Let $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be $2n \times 2n$ matrix written in block form. Prove that S is symplectic if and only if the three equations

$$A^{\mathsf{T}}C = C^{\mathsf{T}}A, \quad B^{\mathsf{T}}D = D^{\mathsf{T}}B, \quad A^{\mathsf{T}}D - C^{\mathsf{T}}B = I$$

hold. ***(11 marks)***

(b) Show that 2×2 matrix is symplectic if and only if its determinant is equal to unity. ***(2 marks)***

(c) Let B be $n \times n$ matrix. When $S = \begin{bmatrix} I & B \\ 0 & I \end{bmatrix}$ is symplectic? ***(2 marks)***

(iii) Consider $\mathbb{R}^4$ with the standard symplectic form Ω. Let W_1, W_2, and W_3 be the vector spaces, $W_1, W_2, W_3 \subseteq \mathbb{R}^4$, defined by

$$W_1 = \{(0,0,0,p_2) \mid p_2 \in \mathbb{R}\},$$

$$W_2 = \{(0,0,p_1,p_2) \mid p_1, p_2 \in \mathbb{R}\},$$

$$W_3 = \{(0,q_2,p_1,p_2) \mid q_1, p_1, p_2 \in \mathbb{R}\}.$$

Find $W_1^{\wedge}$, $W_2^{\wedge}$, and $W_3^{\wedge}$. ***(12 marks)***

2 (i) (a) Define the coordinates (q, p) of a ray in geometrical optics.

(4 marks)

(b) Soppose now that the vertical axis is shifted by distance d along the z-axis. Assuming that the angle φ is small show that the new coordinates of the ray are given by

$$\begin{bmatrix} q' \\ p' \end{bmatrix} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix},$$

where $s = d/n$ is the optical length and n is the refraction index.

(5 marks)

(c) You are given that the coordinates of a ray before and after a parabolic lens are related by

$$\begin{bmatrix} q' \\ p' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -m & 1 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix},$$

where m is the power of the lens. There is an optical system consisting of two lenses a distance s from each other (since the space between the lenses is filled in with the air the optical and physical distances are the same). The power of the left lens is m_1. You are given that the ray incoming in the optical system has coordinates $(q, 0)$ and the ray outgoing of optical system has coordinates $(aq, 0)$, where $a \in \mathbb{R}$, and $q \neq 0$, $a \neq 0$. Find the distance between the lenses, s, and the power of the right lens, m_2. ***(13 marks)***

(ii) Consider $\mathbb{R}^{2n}$ with symplectic form σ. Give the definition of a Lagrangian subspace. Define what it means for two Lagrangian subspaces of $\mathbb{R}^{2n}$ to be transversal. Show that two Lagrangian subspaces, L and L', are transversal if and only if $L \cap L' = \{0\}$. ***(8 marks)***

(iii) (a) Verify that the following subspaces of $(\mathbb{R}^4, \Omega)$

$$L_1 = \text{span}\{(3, 2, -4, 1), (2, 8, 2, -3)\},$$

$$L_2 = \text{span}\{(1, -2, 1, -3), (1, -2, 1, 3)\},$$

are Lagrangian. ***(12 marks)***

(b) Are L_1 and L_2 transversal? ***(8 marks)***

End of Question Paper