The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2019-20**

Galois Theory **1 hour 30 minutes**

This is an open book exam.
Answer ***all*** *questions.*
The submission deadline is 10 am (BST), twenty-four hours after it is released. Late submission will not be considered without extenuating circumstances. It is expected that you will be able to complete this exam in approximately one and a half hour and it is recommended that you submit the work within four hours. You will not be penalised for taking longer, however.
Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working. You will not get full marks if you simply write down output from a computer package.
By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged and that no unfair means have been used.

1 (i) Let L/K and M/K be field extensions.

(a) Let I be a subset of the Gaois group $\mathrm{Gal}(L/K)$. Prove that

$$L^I := \{a \in L \mid \sigma(a) = a \text{ for all } \sigma \in I\}$$

is a subfield of L. ***(4 marks)***

(b) Show that each K-homomorphism $f : L \to M$ is an injection. ***(4 marks)***

(c) Suppose that $[L : K] = [M : K]$. Is every K-homomorphism of fields, $f : L \to M$, a K-isomorphism? Justify your response. ***(4 marks)***

(d) Give a definition of the Galois group $\mathrm{Gal}(L/K)$. Define the operation of multiplication and the inverse in the group $\mathrm{Gal}(L/K)$. ***(5 marks)***

(ii) Let $L = \mathbb{C}(t)$ be the field of rational functions in the variable t over the field $\mathbb{C}$ of complex numbers and σ be a $\mathbb{C}$-automorphism of the field L given by the rule $\sigma(t) = it$ where $i = \sqrt{-1} \in \mathbb{C}$. Let H be the subgroup of $\mathrm{Gal}(L/\mathbb{C})$ generated by σ.

(a) List the elements of the group H. ***(3 marks)***

(b) Let $K = \mathbb{C}(t^4)$ be the subfield of L generated by the element t^4 (K is the field of rational functions in t^4). Find the minimal polynomial of the element $t \in L$ over the field K and show that $[L : K] = 4$. ***(6 marks)***

(c) Prove that $L^H = K$ where $L^H = \{a \in L \mid \sigma(a) = a \text{ for all } \sigma \in H\}$. ***(3 marks)***

(d) Is the field extension L/K Galois? Justify your response. ***(3 marks)***

(e) List the elements of the Galois group $\mathrm{Gal}(L/K)$. ***(3 marks)***

2 (i) Let $K = \mathbb{Q}$ be the field of rational numbers and L be the splitting field of the polynomial $f(x) = x^3 - 7$.

(a) Show that $L = \mathbb{Q}(\alpha, \omega)$ where $\alpha = \sqrt[3]{7}$ and $\omega = e^{\frac{2\pi i}{3}}$. ***(4 marks)***

(b) Let $M = \mathbb{Q}(\alpha)/\mathbb{Q}$. Show that $[M : \mathbb{Q}] = 3$, $[L : M] = 2$ and $[L : \mathbb{Q}] = 6$. ***(6 marks)***

(c) Prove that the field extensions $L/\mathbb{Q}$ and L/M are Galois and find the orders of the groups $\mathrm{Gal}(L/\mathbb{Q})$ and $\mathrm{Gal}(L/M)$. ***(5 marks)***

End of Question Paper