Data provided: Supplementary Formula Sheet

MAS452/MAS6052

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2019–2020**

Stochastic Processes and Financial Mathematics **1.5 hours (nominal)**

*Candidates should attempt **ALL** questions.*
The maximum marks for the various parts of the questions are indicated.
The paper will be marked out of 50.

This is an open book exam.

The submission deadline is 10 am (BST), twenty-four hours after it is released. Late submission will not be considered without extenuating circumstances. It is expected that you will be able to complete this exam in approximately 1.5 hours and it is recommended that you submit the work within four hours. You will not be penalised for taking longer, however.

You may use a calculator, but to gain full marks you will need to show your working. You will not get full marks if you simply write down output from a computer package.

By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools, other than material provided for this module, is cited and acknowledged and that no unfair means have been used.

1 (a) Let $\Omega = \{1,2,3,4,5\}$ and let

$$\mathcal{F}_1 = \{\emptyset, \{1,3\}, \{2,4,5\}, \Omega\}$$
$$\mathcal{F}_2 = \{\emptyset, \{1,2\}, \{3,4\}, \{5\}, \Omega\}$$
$$\mathcal{F}_3 = \{\emptyset, \{1\}, \{2,3\}, \{1,2,3\}, \Omega\}$$
$$\mathcal{F}_4 = \sigma(\{1,2,4\}, \{1\}, \{3,5\})$$

Note that each $\mathcal{F}_i$ is a set of subsets of Ω.

(i) For each $\mathcal{F}_i$ ($i = 1, \ldots, 4$) state, without proof, whether $\mathcal{F}_i$ is a σ-field.

(ii) For each of the $\mathcal{F}_i$ that are *not* σ-fields, briefly explain why not.

(6 marks)

2 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Let X and Y be random variables, and suppose that $Y = \mathbb{E}[X|\mathcal{G}]$.

(a) Show that:

(i) $\mathbb{E}[X] = \mathbb{E}[Y]$.

(ii) $\mathbb{E}[(X-Y)^2] = \mathbb{E}[X^2] - \mathbb{E}[Y^2]$.

(iii) $\mathrm{var}(Y) \le \mathrm{var}(X)$.

(7 marks)

(b) Consider the following four statements.

1. *"Conditional expectation increases randomness, which increases variance."*
2. *"Conditional expectation reduces randomness, which reduces variance."*
3. *"Conditional expectation makes random variables smaller, which reduces their variance."*
4. *"Conditional expectation makes random variables larger, which increases their variance."*

Answer the following questions, about statements 1-4.

(i) Which one of the four statements most accurately describes what you have seen in part (a)? Justification is not required.

(ii) Are any of the other three statements true? If so, which ones? Justification is not required.

(2 marks)

3 This question concerns the binomial model, in discrete time, with two assets, cash and stock.

Suppose that we have $T = 2$ steps of time, and let the parameters of the model be $p_u = 0.7$, $p_d = 0.3$, $u = 1.5$, $d = 0.75$, $r = 0.25$ and $s = 240$.

Consider the contingent claim

$$\Phi(S_T) = \max(5S_T - 900, 0)$$

Draw a recombining tree of the stock price process at time $t = 0, 1, 2$. Annotate your tree to show the arbitrage free price for this contingent claim, at each node, along with a portfolio strategy that hedges $\Phi(S_T)$. ***(10 marks)***

4 Let $p \in [0, 1]$. Let $(X_i)_{i=1}^{\infty}$ and $(Y_i)_{i=1}^{\infty}$ be sequences of independent, identically distributed random variables with distribution

$$\mathbb{P}[X_i = 1] = \mathbb{P}[X_i = -1] = \frac{1}{2}, \qquad \mathbb{P}[Y_i = 0] = p, \qquad \mathbb{P}[Y_i = 1] = 1 - p.$$

We assume that X_i and Y_j are independent for all i, j.

Define a stochastic process $(S_n)_{n=0}^{\infty}$ by setting $S_0 = 1$ and, for $n \geq 0$,

$$S_{n+1} = \begin{cases} S_n + X_{n+1} & \text{if } Y_n = 0 \\ S_n & \text{if } Y_n = 1. \end{cases}$$

(a) Show that (S_n) is a martingale, with respect to a filtration $(\mathcal{F}_n)$ that you should define. ***(6 marks)***

(b) Define $T_0 = 0$ and $T_{n+1} = \inf\{m > T_n \,;\, Y_m = 0\}$.

(i) Show that T_n is a stopping time, for all n.

Hint: You may use that T is a stopping time with respect to the filtration $(\mathcal{G}_m)$ if and only if $\{T = m\} \in \mathcal{G}_m$ for all $m \in \mathbb{N}$.

(ii) Show that the process $M_n = S_{T_n}$ is a symmetric random walk with $|M_{n+1} - M_n| = 1$.

(6 marks)

5 This question concerns the Black-Scholes model, in continuous time.

A brief summary of the Black-Scholes model, and associated notation, can be found on the supplementary formula sheet.

Set $r = 0$ and $\sigma = 1$. Consider the contingent claim

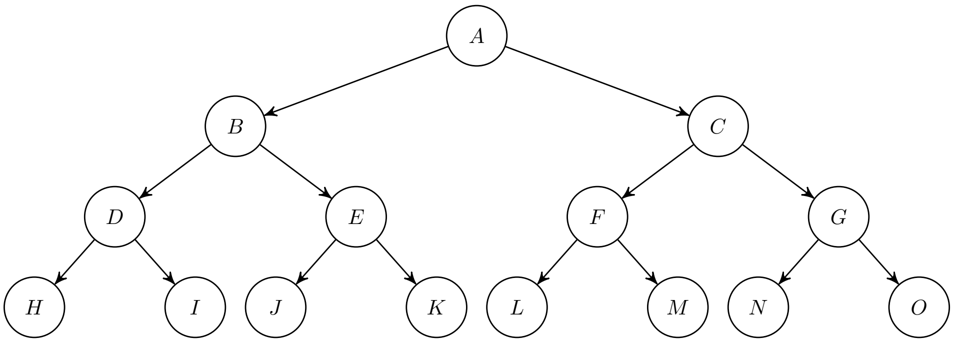
$$\Phi(S_T) = (\log S_T)^2.$$

Find the value, at time $t \in [0, T)$, of the contingent claim $\Phi(S_T)$. ***(6 marks)***

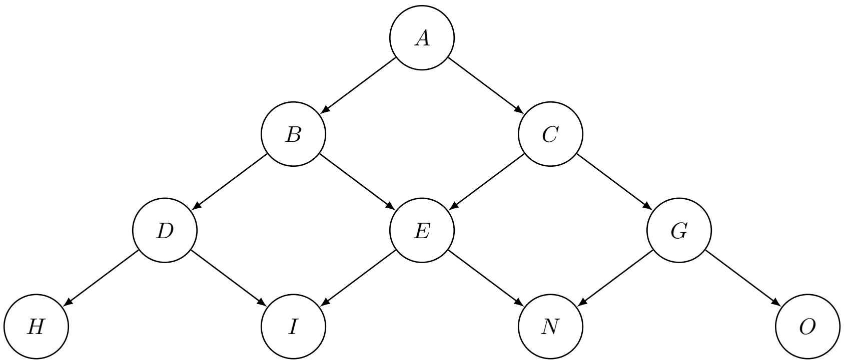
6 Consider the Gai-Kapadia model of debt contagion, on two different financial networks.

A brief summary of the Gai-Kapadia model, and associated notation, can be found on the supplementary formula sheet.

Network 1:



Network 2:



In both networks, we assume the same contagion probabilities, $\eta_j = p \in (0, 1)$.

Suppose that, in both networks, node A fails, and that all other nodes are initially healthy.

(a) (i) Calculate the probability that node E fails, in Network 1.

(ii) Calculate the probability that node E fails, in Network 2.

(iii) Show that node I is more likely to fail in Network 2 than Network 1.

(4 marks)

(b) Is the following statement true or false? Justify your answer.

The expected proportion of failed nodes is strictly greater in Network 1 than in Network 2.

(3 marks)

End of Question Paper

MAS352/452/6052 – Formula Sheet – Part One

Where not explicitly specified, the notation used matches that within the typed lecture notes.

Modes of convergence

- $X_n \stackrel{d}{\to} X \;\Leftrightarrow\; \lim_{n\to\infty} \mathbb{P}[X_n \le x] = \mathbb{P}[X \le x]$ whenever $\mathbb{P}[X \le x]$ is continuous at $x \in \mathbb{R}$.
- $X_n \stackrel{\mathbb{P}}{\to} X \;\Leftrightarrow\; \lim_{n\to\infty} \mathbb{P}[|X_n - X| > a] = 0$ for every $a > 0$.
- $X_n \stackrel{a.s.}{\to} X \;\Leftrightarrow\; \mathbb{P}\left[X_n \to X \text{ as } n \to \infty\right] = 1$.
- $X_n \stackrel{L^p}{\to} X \;\Leftrightarrow\; \mathbb{E}\left[|X_n - X|^p\right] \to 0$ as $n \to \infty$.

The binomial model and the one-period model

The binomial model is parametrized by the deterministic constants r (discrete interest rate), p_u and p_d (probabilities of stock price increase/decrease), u and d (factors of stock price increase/decrease), and s (initial stock price).

The value of x in cash, held at time t, will become $x(1+r)$ at time $t+1$.

The value of a unit of stock S_t, at time t, satisfies $S_{t+1} = Z_t S_t$, where $\mathbb{P}[Z_t = u] = p_u$ and $\mathbb{P}[Z_t = d] = p_d$, with initial value $S_0 = s$.

When $d < 1 + r < u$, the risk-neutral probabilities are given by

$$q_u = \frac{(1+r) - d}{u - d}, \qquad q_d = \frac{u - (1+r)}{u - d}.$$

The binomial model has discrete time $t = 0, 1, 2, \ldots, T$. The case $T = 1$ is known as the one-period model.

Conditions for the optional stopping theorem (MAS452/6052 only)

The optional stopping theorem, for a martingale M_n and a stopping time T, holds if any one of the following conditions is fulfilled:

(a) T is bounded.

(b) M_n is bounded and $\mathbb{P}[T < \infty] = 1$.

(c) $\mathbb{E}[T] < \infty$ and there exists $c \in \mathbb{R}$ such that $|M_n - M_{n-1}| \le c$ for all n.

MAS352/452/6052 – Formula Sheet – Part Two

Where not explicitly specified, the notation used matches that within the typed lecture notes.

The normal distribution

$Z \sim N(\mu, \sigma^2)$ has probability density function $f_Z(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$.

Moments: $\mathbb{E}[Z] = \mu$, $\mathbb{E}[Z^2] = \sigma^2 + \mu^2$, $\mathbb{E}[e^Z] = e^{\mu + \frac{1}{2}\sigma^2}$.

Ito's formula

For an Ito process X_t with stochastic differential $dX_t = F_t\,dt + G_t\,dB_t$, and a suitably differentiable function $f(t,x)$, it holds that

$$dZ_t = \left\{ \frac{\partial f}{\partial t}(t, X_t) + F_t \frac{\partial f}{\partial x}(t, X_t) + \frac{1}{2} G_t^2 \frac{\partial^2 f}{\partial x^2}(t, X_t) \right\} dt + G_t \frac{\partial f}{\partial x}(t, X_t)\,dB_t$$

where $Z_t = f(t, X_t)$.

Geometric Brownian motion

For deterministic constants $\alpha, \sigma \in \mathbb{R}$, and $u \in [t, T]$ the solution to the stochastic differential equation $dX_u = \alpha X_u\,dt + \sigma X_u\,dB_u$ satisfies

$$X_T = X_t e^{(\alpha - \frac{1}{2}\sigma^2)(T-t) + \sigma(B_T - B_t)}.$$

The Feynman-Kac formula

Suppose that $F(t,x)$, for $t \in [0,T]$ and $x \in \mathbb{R}$, satisfies

$$\begin{aligned} \frac{\partial F}{\partial t}(t,x) + \alpha(t,x)\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}\beta(t,x)^2 \frac{\partial^2 F}{\partial x^2}(t,x) - rF(t,x) &= 0 \\ F(T,x) &= \Phi(x). \end{aligned}$$

If X_u satisfies $dX_u = \alpha(u, X_u)\,dt + \beta(u, X_u)\,dB_u$, then

$$F(t,x) = e^{-r(T-t)} \mathbb{E}_{t,x}\left[\Phi(X_T)\right].$$

The Black-Scholes model

The Black-Scholes model is parametrized by the deterministic constants r (continuous interest rate), μ (stock price drift) and σ (stock price volatility).

The value of a unit of cash C_t satisfies $dC_t = rC_t\,dt$, with initial value $C_0 = 1$.

The value of a unit of stock S_t satisfies $dS_t = \mu S_t\,dt + \sigma S_t\,dB_t$, with initial value S_0.

At time $t \in [0, T]$, the price $F(t, S_t)$ of a contingent claim $\Phi(S_T)$ (satisfying $\mathbb{E}^{\mathbb{Q}}[\Phi(S_T)] < \infty$) with exercise date $T > 0$ satisfies the Black-Scholes PDE:

$$\frac{\partial F}{\partial t}(t,s) + rs\frac{\partial F}{\partial s}(t,s) + \frac{1}{2}s^2\sigma^2\frac{\partial^2 F}{\partial s^2}(t,s) - rF(t,s) = 0,$$
$$F(T,s) = \Phi(s).$$

The unique solution F satisfies

$$F(t, S_t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\Phi(S_T) \,|\, \mathcal{F}_t]$$

for all $t \in [0, T]$. Here, the 'risk-neutral world' $\mathbb{Q}$ is the probability measure under which S_t satisfies

$$dS_t = rS_t\,dt + \sigma S_t\,dB_t.$$

The Gai-Kapadia model of debt contagion (MAS452/6052 only)

A financial network consists of banks and loans, represented respectively as the vertices V and (directed) edges E of a graph G. An edge from vertex X to vertex Y represents a loan owed by bank X to bank Y.

Each loan has two possible states: healthy, or defaulted. Each bank has two possible states: healthy, or failed. Initially, all banks are assumed to be healthy, and all loans between all banks are assumed to be healthy.

Given a sequence of contagion probabilities $\eta_j \in [0, 1]$, we define a model of debt contagion by assuming that:

(†) For any bank X, with in-degree j if, at any point, X is healthy and one of the loans owed to X becomes defaulted, then with probability η_j the bank X fails, independently of all else. All loans owed by bank X then become defaulted.

Starting from some set of newly defaulted loans, the assumption (†) is applied iteratively until no more loans default.