The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2019–2020**

Financial Mathematics **Nominal length: 1 hour and 30 minutes**

This is an open book exam.
*Answer **all** questions.*

The submission deadline is 10 am (BST), twenty-four hours after it is released. Late submission will not be considered without extenuating circumstances. It is expected that you will be able to complete this exam in approximately one and a half hour and it is recommended that you submit the work within four hours. You will not be penalised for taking longer, however.

Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working. You will not get full marks if you simply write down output from a computer package.

By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged and that no unfair means have been used.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 (i) Consider the following five bonds with face value of £100:

Time to maturity (in years)	Annual interest (paid every 6 months)	Bond price (in £)
0.25	0	99.0
0.5	0	97.8
1.0	0	95.5
1.5	8%	104.5

(a) Find the 0.25, 0.5 and 1-year spot interest rates. ***(6 marks)***

(b) Use the bootstrap method to find the 1.5 spot interest rate. ***(4 marks)***

(ii) The price of a stock which pays no dividends is currently £100. Over each of the next two one-year periods the stock price will either double or halve. Suppose that all interest rates are constant and equal to 10%.

(a) Use a binomial tree to find the price of a two-year American put option on this stock with strike price £150. ***(11 marks)***

(b) Describe all circumstances in which a rational investor should exercise the option. ***(4 marks)***

(iii) Suppose that c_1, c_2, and c_3 are the prices of European call options with strike prices X_1, X_2, and X_3 respectively, where $X_3 > X_2 > X_1$ and $X_3 - X_2 = X_2 - X_1$. We are told that all options have the same expiration date. Show that

$$c_2 \leq \frac{1}{2}(c_1 + c_3).$$ ***(12 marks)***

(iv) Consider a European call option on an asset with spot price S, expiration time in one year, and strike price S. You are told that a year from now the price of the asset will have either risen by a factor of u ($u > 1$) or will have decreased by a factor of $\frac{1}{u}$. If the interest rate on one-year deposits is $r > 0$, show that the price of the European call option is an increasing function of the variable u. ***(13 marks)***

2 (i) In this question we consider a European call option and a European put option, both with the same underlying stock, the same strike price X and the same maturity time $T > 0$. For any time $0 \leq t \leq T$, let $S_t = S$ denote the spot price of the underlying stock, let $c(S, t)$ be the price of European call and let $p(S, t)$ be the price of European put option at time t. Assume that all spot interest rates are constant and equal to r. Assume also that the underlying stock price follows the Ito process

$$dS = rSdt + \sigma SdB$$

(a) Verify that $f(S, t) = Xe^{-r(T-t)} - S$ is a solution of the Black-Scholes partial differential equation

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf.$$

(4 marks)

(b) Deduce that $g(S, t) = Xe^{-r(T-t)} - S + c(S, t) - p(S, t)$ is also a solution of the Black-Scholes partial differential equation. ***(3 marks)***

(c) By considering the case $t = T$, show that (b) implies that

$$c(S, t) + Xe^{-r(T-t)} = p(S, t) + S.$$

(8 marks)

(ii) For any time $0 \leq t \leq T$, let $S_t = S$ denote the spot price of a stock. Assume that all spot interest rates are constant and equal to r. We also assume that the stock price S_t follows geometric Brownian motion $dS = rSdt + \sigma SdB$.

(a) Show that the process $S_t^n, n \geq 2$ also follows geometric Brownian motion, and find its drift. ***(6 marks)***

(b) Now consider a derivative, on the same stock, that pays off S_T^n at time T, where S_T is the stock price at that time. If you are told that the price of the derivative at time t ($t \leq T$) has the form

$$h(t)S_t^n,$$

where S_t is the stock price at time t, h is a function only of t, and that the price of the derivative satisfies the Black-Scholes equation,

(α) find the differential equation that $h(t)$ satisfies. ***(5 marks)***

(β) find the value of $h(T)$, which plays the role of a "boundary" condition, for the above differential equation ***(4 marks)***

 Question 2 continued on next page

2 (continued)

(iii) Consider a market with risk free interest rate $r_B = 5\%$ and two risky investments A and B. We are given the following data

Investment	Expected return	Standard deviation of return
A	10%	10%
B	15%	20%

and we are also told that the correlation between the returns of A and B is $\rho = 0.5$.

We assume that the Capital Asset Pricing Model holds.

(a) Explain why the expected returns of the market portfolio must exceed the risk free return. ***(2 marks)***

(b) Use the fact that the market portfolio is the unique portfolio which maximises

$$\frac{r_P - r_B}{\sigma_P},$$

as P ranges over all portfolios consisting entirely of risky investments to find the market portfolio in the market described above. ***(8 marks)***

(c) As the fund manager of Successful Investments you are asked to invest £100,000,000 in a portfolio consisting of the investments A and B above, and the risk free investment. The portfolio should have an expected return of 12% and the lowest possible standard deviation of returns. Describe that portfolio. ***(10 marks)***

End of Question Paper