The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2019–2020**

Mathematical Methods And Modelling Of Natural Systems **1 hour**

This is an open book exam.
Answer ***both*** *questions. The marks awarded to each section of question are shown in italics. The total mark for the paper is 50.*
The submission deadline is 10 am (BST), twenty-four hours after it is released. Late submission will not be considered without extenuating circumstances. It is expected that you will be able to complete this exam in approximately one hour and it is recommended that you submit the work within four hours. You will not be penalised for taking longer, however.
Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working. You will not get full marks if you simply write down output from a computer package.
By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged and that no unfair means have been used.

1 The scalar field $\phi(x,t)$ satisfies the wave equation

$$\frac{\partial^2\phi}{\partial t^2} = c^2\frac{\partial^2\phi}{\partial x^2} \qquad -\infty < x < \infty, \quad t > 0,$$

where c is a positive constant. The initial conditions at $t = 0$ are $\phi = f(x)$ and $\dfrac{\partial\phi}{\partial t} = g(x)$.

The Fourier transform of $\phi(x,t)$ with respect to x is defined by

$$\hat{\phi}(k,t) = \int_{-\infty}^{\infty} e^{ikx}\,\phi(x,t)\,dx.$$

Show that

$$\hat{\phi}(k,t) = \hat{f}(k)\cos kct + \frac{\hat{g}(k)}{kc}\sin kct,$$

where $\hat{f}(k)$ and $\hat{g}(k)$ are the Fourier transforms with respect to x of $f(x)$ and $g(x)$, respectively. ***(9 marks)***

Hence find $\phi(x,t)$ when

(a) $f(x) = 0, \quad g(x) = \lambda c\sin\lambda x$ (where λ is a constant), ***(7 marks)***

[You may assume that
$$\int_{-\infty}^{\infty} e^{ikx}\,dx = 2\pi\delta(k).$$]

(b) $f(x) = \dfrac{1}{2\sqrt{\pi}}e^{-x^2/4}, \quad g(x) = 0.$ ***(9 marks)***

[You may assume that
$$\int_{-\infty}^{\infty} e^{-(x-a)^2}\,dx = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi},$$
where a may be complex, but does not depend on x.]

2 The function $y(x)$ satisfies the ordinary differential equation

$$y'' - 4y = \ln(1+x^2) \qquad 0 \leqslant x \leqslant 2, \tag{1}$$

with the boundary conditions

$$y = 0 \quad \text{at} \quad x = 0 \quad \text{and at} \quad x = 2.$$

(a) Find the independent solutions of

$$y'' - 4y = 0.$$

(3 marks)

(b) Given that Green's function $G(x;\xi)$ for the boundary-value problem given at the beginning of the question is continuous at $x = \xi$, and that $\partial G/\partial x$ has a discontinuity of size 1 at $x = \xi$, show that

$$G(x;\xi) = \begin{cases} \dfrac{\sinh 2(\xi - 2)\ \sinh 2x}{2\sinh 4} & 0 \leqslant x < \xi, \\[2ex] \dfrac{\sinh 2\xi\ \sinh 2(x-2)}{2\sinh 4} & \xi < x \leqslant 2. \end{cases}$$

(14 marks)

(c) Use Green's function to write down the solution to equation (1) and the boundary conditions given at the beginning of the question (do NOT attempt the ξ integrals). ***(3 marks)***

Use this to find $y'(x)$, and hence to show that

$$y'(2) = \frac{1}{\sinh 4}\int_0^2 \sinh 2\xi\ \ln(1+\xi^2)\, d\xi.$$

(5 marks)

End of Question Paper