The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring semester 2019-2020**

Magnetohydrodynamics **1 Hour**

This is an open book exam.
*Answer **all** questions.*
The submission deadline is 10 am (BST), twenty-four hours after it is released. Late submission will not be considered without extenuating circumstances. It is expected that you will be able to complete this exam in approximately one hour and it is recommended that you submit the work within four hours. You will not be penalised for taking longer, however.
Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working. You will not get full marks if you simply write down output from a computer package.
By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged and that no unfair means have been used.

1 (i) In the corona active regions, the estimate of the density is $\rho = 1.67 \times 10^{-11}\text{kgm}^{-3}$, the magnitude of the magnetic field is $B = 10^{-2}$Tesla, the temperature is $T = 2 \times 10^6$K, the plasma pressure is $p = 0.55$Pa, the magnetic diffusivity is $\eta = 1\text{m}^2\text{s}^{-1}$, the velocity scale is $v = 3 \times 10^4\text{ms}^{-1}$, and the length scale is $L = 2 \times 10^7$m. Given that the magnetic permeability $\mu = 4\pi \times 10^{-7}$, estimate the plasma β, the magnetic Reynolds number R_m, the diffusion time scale, and the Alfvén speed, keeping up to three significant figures. ***(4 marks)***

(ii) Let $\mathbf{A}$ be a vector potential of the magnetic field $\mathbf{B}$, defined by $\mathbf{B} = \nabla \times \mathbf{A}$. It is given that $\mathbf{A}$ satisfies the following equation

$$\frac{\partial \mathbf{A}}{\partial t} = \boldsymbol{v} \times (\nabla \times \mathbf{A}), \tag{1}$$

where $\boldsymbol{v}$ is the velocity, which is assumed to be incompressible. Without using the Alfvén theorem, show that

$$\frac{d}{dt} \oint_{C(t)} \mathbf{A} \cdot d\boldsymbol{\ell} = 0, \tag{2}$$

where $C(t)$ is the bounding curve of a material surface and $d\boldsymbol{\ell}$ is a line element on $C(t)$ in the anti-clockwise direction. ***(10 marks)***

(iii) Consider a zero-β plasma with uniform density ρ_0 in a uniform magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$. We will neglect gravity and magnetic diffusion and assume the plasma is in equilibrium. You are given that the linearised equations for the velocity perturbation $\boldsymbol{v}$ and the magnetic field perturbation $\mathbf{B}_1$ are given by

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} = -\frac{B_0}{\mu}\nabla B_{1z} + \frac{B_0}{\mu}\partial_z \mathbf{B}_1, \tag{3}$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = B_0 \partial_z \boldsymbol{v} - B_0 \hat{\mathbf{z}} \nabla \cdot \boldsymbol{v}, \tag{4}$$

where μ is the magnetic permeability. By considering plane wave perturbations in the form of

$$(\boldsymbol{v}, \mathbf{B}_1) = (\hat{\boldsymbol{v}}, \hat{\mathbf{B}}_1) \exp[i(k_x x + k_z z - \omega t)], \tag{5}$$

derive the dispersion relations from equations (3) and (4) (written in terms of the Alfvén speed where possible). ***(11 marks)***

End of Question Paper

Formulae Sheet

1. $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = (\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A} = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = (\mathbf{C} \times \mathbf{A}) \cdot \mathbf{B}$
2. $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{C} \times \mathbf{B}) \times \mathbf{A} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$
3. $\boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla} g + g\boldsymbol{\nabla} f$
4. $\boldsymbol{\nabla}(a/b) = (1/b)\boldsymbol{\nabla} a - (a/b^2)\boldsymbol{\nabla} b$
5. $\boldsymbol{\nabla} \cdot (f\mathbf{A}) = (\boldsymbol{\nabla} f) \cdot \mathbf{A} + f(\boldsymbol{\nabla} \cdot \mathbf{A})$
6. $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} f = \nabla^2 f$
7. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = 0$
8. $\boldsymbol{\nabla} \times (f\mathbf{A}) = (\boldsymbol{\nabla} f) \times \mathbf{A} + f(\boldsymbol{\nabla} \times \mathbf{A})$
9. $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = 0$
10. $\boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A} - (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} + (\boldsymbol{\nabla} \cdot \mathbf{B})\mathbf{A} - (\boldsymbol{\nabla} \cdot \mathbf{A})\mathbf{B}$
11. $\nabla^2 \mathbf{A} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{A}$
12. $\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) - \mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{B})$
13. $\mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{B}) = (\boldsymbol{\nabla} \mathbf{B}) \cdot \mathbf{A} - (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B}$
14. $\boldsymbol{\nabla}(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A} + (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} + \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{A}) + \mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{B})$
15. $\boldsymbol{\nabla} \cdot (\mathbf{A}\mathbf{B}) = (\boldsymbol{\nabla} \cdot \mathbf{A})\mathbf{B} + (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B}$