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The University Of Sheffield.

AER201

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2020–21**

Mathematics for Aerospace Engineers

This is an open book exam.

*Answer **all** questions.*

*This exam starts at 10am (GMT), and you must submit your work within two and a half hours (that is, by 12:30pm (GMT)). **Late submission will not be considered without extenuating circumstances.***

Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working.

By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.

Total marks available: 40.

1 Items from a production line pass a quality control check with probability 0.8. Checks are conducted on batches of 25 items and, although a batch is composed of items produced at about the same time, you may regard the individual items in each batch as independent.

(i) Let X be the number of items failing the test in a randomly selected batch. Explain why

$$X \sim Bin(25, 0.2)$$

is a suitable model. **(1 mark)**

(ii) Explain, with reasons, whether you think the assumption of independence of items within a batch is realistic in practice. **(1 mark)**

(iii) Let T be the total number of failing items in 100 independent batches. Write down a Normal distribution which approximates the distribution of T, naming the result that tells you which Normal distribution to use. Hence give a MatLab command to evaluate the approximate probability that T exceeds 520. **(4 marks)**

(iv) Items which fail the test may be reworked at a fraction K of the original production cost. K follows the distribution with probability density function

$$f(k) = tk(1-k) \qquad k \in [0, 1]$$

where t is a constant. Find the value of t and hence, or otherwise, show that the expected value of K is 0.5. **(4 marks)**

2 Find the Fourier transform $F(s)$ for the function

$$f(t) = \begin{cases} a - |t| & \text{if } |t| \leq a, \\ 0 & \text{if } |t| > a, \end{cases}$$

where $a > 0$. **(10 marks)**

3 (i) Sketch the domain D where

$$D = \{(x, y, z) \in \mathbb{R}^3 \,|\, x, y, z \geq 0, x + y + z \leq \pi\}.$$

(5 marks)

(ii) Evaluate the integral

$$I = \iiint_D \sin(x + y + z) dx dy dz.$$

(15 marks)

End of Question Paper

AER201 FORMULA SHEET

Standard Probability Distributions:

Name	**Applications**	**Notation**	**pmf** or **pdf**	$E(X)$	$\text{Var}(X)$
Bernoulli trial	Expt with two outcomes. Coins, constituent of more complex distributions. $X \equiv$ no. successes	$Bernoulli(p)$	$p_X(1) = p$, $p_X(0) = 1 - p$ $p\epsilon[0,1]$	p	$p(1-p)$
Binomial	$X \equiv$ no. successes in n ind. Bernoulli trials Sampling with replacement	$Bin(n,p)$	$p_X(x) = \frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}$ $x = 0, 1, 2, ..., n$ $p\epsilon[0,1]$	np	$np(1-p)$
Geometric	$X \equiv$ total no. of trials until 1st success in sequence of ind. Bernoulli trials Waiting times	Geo(p)	$p_X(x) = (1-p)^{x-1}p$ $x = 1, 2, ...$ $p\epsilon[0,1]$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson	Counting events occurring 'at random' in space or time	$Po(\lambda)$	$p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}$ $x = 0, 1, 2, ...$ $\lambda > 0$	λ	λ
Multinomial	Generalization of Binomial to > 2 categories	$multinomial(n; p_1, ..., p_k)$	$p_X(x_1, \ldots, x_k) =$ $\frac{n!}{x_1!x_2!\cdots x_k!}p_1^{x_1}p_2^{x_2}\ldots p_k^{x_k}$		
Uniform	Rounding errors $Un(-\frac{1}{2}, \frac{1}{2})$	$Un(a,b)$	$f_X(x) = \frac{1}{b-a}$ $x \in [a,b]$ $a < b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential	Lifetimes of non-ageing items	$Exp(\lambda)$	$f_X(x) = \lambda e^{-\lambda x}$ $x > 0$ $\lambda > 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Normal	Empirically, and theoretically via CLT, a good model in many situations	$N(\mu, \sigma^2)$	$f_X(x) =$ $\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$ $x \in (-\infty, \infty)$	μ	σ^2
Multivariate Normal	Empirically a good model in many situations	$N_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$	$f_X(x_1, x_2, ..., x_k) =$ $\frac{1}{\sqrt{(2\pi)^k\lvert\Sigma\rvert}} e^{\frac{-(\mathbf{X}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu})}{2}}$	$\boldsymbol{\mu}$	$\boldsymbol{\Sigma}$

Bayes' Theorem:

Suppose we have two events E and F within a sample space S, then

$$P(E|F) = \frac{P(F|E)P(E)}{P(F)}$$

Central Limit Theorem:

Let $X_1, X_2, \ldots, X_n$ be a sequence of i.i.d random variables, each with mean μ and variance σ^2, then for large n we have, approximately,

$$\bar{X}(n) \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

or, equivalently,

$$\Sigma_{i=1}^{n} X_i \sim N\left(n\mu, n\sigma^2\right)$$

Laplace transform:

The Laplace transform of a function $f(t)$ is given by:

$$\mathcal{L}\{f(t)\}(s) := \int_0^\infty e^{-st} f(t) dt.$$

Properties of the Laplace transform: $\mathcal{L}\{f(t)\} = F(s)$ in the following table.

$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	linearity
$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$	differentiation w.r.t. t
$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$	second differentiation w.r.t. t
$\mathcal{L}\{e^{-kt}f(t)\} = F(k+s)$	frequency shift
$\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}F(s) \qquad (\text{for } a > 0)$	time shift
$\mathcal{L}\{f(at)\} = \frac{1}{a}F(\frac{s}{a}) \qquad (\text{for } a > 0)$	scaling
$\mathcal{L}\{f * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$ (for $f(t), g(t)$ causal)	convolution

Table of standard Laplace transforms:

$f(t)$	$\mathcal{L}\{f(t)\}(s)$	Region of validity
$t^n \qquad (\text{for } n \geq 0)$	$\frac{n!}{s^{n+1}}$	$Re(s) > 0$
$\sin(kt)$	$\frac{k}{s^2+k^2}$	$Re(s) > 0$
$\cos(kt)$	$\frac{s}{s^2+k^2}$	$Re(s) > 0$
$H(t-T) \qquad (\text{for } T \geq 0)$	$\frac{e^{-sT}}{s}$	$Re(s) > 0$
$\delta(t-T) \qquad (\text{for } T \geq 0)$	e^{-sT}	$s \in \mathbb{C}$

Fourier transform:

The Fourier transform and inverse Fourier transforms are given by:

$$\mathcal{F}\{f(t)\}(\omega) = F(\omega) := \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt, \qquad f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega.$$

Properties of the Fourier transform: $\mathcal{F}\{f(t)\} = F(\omega)$ in the following table:

$\mathcal{F}\{e^{j\theta t}f(t)\} = F(\omega - \theta)$	frequency shift
$\mathcal{F}\{f(t-T)\} = e^{-j\omega T}F(\omega)$	time shift
$\mathcal{F}\{f^{(n)}(t)\} = (j\omega)^n F(\omega)$	differentiation
$\mathcal{F}\{F(t)\} = 2\pi f(-\omega)$	symmetry
$\mathcal{F}\{f(at)\} = \frac{1}{\lvert a\rvert}F(\frac{\omega}{a})$	scaling
$\mathcal{F}\{f * g(t)\} = \mathcal{F}\{f(t)\}\mathcal{F}\{g(t)\}$	convolution

Table of standard Fourier transforms:

$f(t)$	$\mathcal{F}\{f(t)\}(\omega)$
$e^{-a\lvert t\rvert}$ (for $a > 0$)	$\frac{2a}{a^2+\omega^2}$
$\text{rect}_T(t)$	$\text{sinc}(\frac{T\omega}{2})$
1	$2\pi\delta(\omega)$

Fourier series:

The Fourier series of a periodic function $f(t)$ with fundamental period T is given by

$$S[f] = \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n\cos(\omega_n t) + b_n\sin(\omega_n t)\Big)$$

where

$$\omega_n = \frac{2\pi n}{T}, \qquad a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(\omega_n t)dt, \qquad b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(\omega_n t)dt.$$

Coordinate systems:

Cylindrical polar coordinates

$$(x, y, z) = (r\cos(\theta), r\sin(\theta), z)$$

$$(r, \theta, z) = \left(\sqrt{x^2+y^2}, \arctan(\tfrac{y}{x}), z\right)$$

$$dV = r dr d\theta dz.$$

Spherical polar coordinates

$$(x, y, z) = (\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))$$

$$(\rho, \theta, \phi) = \left(\sqrt{x^2+y^2+z^2}, \arctan(\tfrac{y}{x}), \arccos(\tfrac{z}{\rho})\right)$$

$$dV = \rho^2\sin(\phi)d\rho d\phi d\theta.$$