MAS248 Exam questions 2020-21

1. A random variable X has mean 5 and variance 16. If $Y = (X-3)^2$, find $E[Y]$ (6 marks)

2. A random variable X has probability density function $f(x) = cx(6-x)^2$ for $0 \leq x \leq 6$, where c is a constant. $f(x) = 0$ for values of x outside this range. The mode of X is defined to be the value of X for which the probability density function is a maximum. Find the mode of X. (8 marks)

3. The probability density function of a continuous uniform distribution is given by $f(x) = 1/(b-a)$ for $a \leq x \leq b$, where a and b are constants, and $f(x) = 0$ outside this interval. Let X be a uniformly distributed random variable over the interval $[a, b] = [-10, 2]$. What is the variance of X? (6 marks)

4. The function $f(x) = x^3 + 2x^2 - x - 10 = 0$ has a root in the interval $[1, 2]$. How many iterations of the bisection method are required to achieve an approximation to the root to an accuracy of 10^{-3}? (8 marks)

5. The function $f(x) = x^3 - x - 1$ has a root near $x = 1$. Use the Newton-Raphson method to calculate the root correct to 4 decimal places, using an initial guess of $x_0 = 1.0$. (7 marks)

6. The Fourier series of the function $f(x) = x \sin x$ for $0 < x \leq 2\pi$, where $f(x + 2\pi) = f(x)$ for all x, can be expressed in the form $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$. What is the value of the Fourier coefficient a_0? (9 marks)

7. The Fourier series of the function $f(x)$, where $f(x) = 0$ for $-2 < x \leq 0$, $f(x) = x$ for $0 < x \leq 2$ and $f(x+4) = f(x)$ for all x, can be expressed in the form $\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$, where L is half the period of $f(x)$. Find, correct to two decimal places, the value of the

Fourier coefficient b_3. (12 marks)

8. Consider the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

that satisfies the initial conditions

$$u(x,0) = \sin^2 x$$

and

$$\frac{\partial u}{\partial t}(x,0) = \cos x.$$

Which of the following expressions for $u(x,t)$ satisfy this wave equation together with the initial conditions? Select all that apply. (10 marks)

(a) $u(x,t) = \frac{1}{2}\Big(\sin^2(x+t) + \sin^2(x-t) + \sin(x+t) - \sin(x-t)\Big)$

(b) $u(x,t) = \frac{1}{2}\Big(\sin^2(x+t) + \sin^2(x-t)\Big) + \sin 2x$

(c) $u(x,t) = \frac{1}{2}\Big(\sin^2(x+t) + \sin^2(x-t)\Big) + 2\sin t\cos x$

(d) $u(x,t) = (\sin x \cos t)^2 + (\sin t \cos x)^2 + \sin t \cos x$

9. Let $f(r)$ be a scalar field given by $f(r) = \frac{1}{r^n}$ where n is an integer and $r = \sqrt{x^2+y^2+z^2}$. Which one of the following terms denotes $\nabla^2 f(r)$? (9 marks)

(a) $\frac{n(n+5)}{r^{n+2}}$

(b) $\frac{n}{r^{n+2}}$

(c) $-\frac{1}{r^{n+2}}$

(d) $\frac{n(n-1)}{r^{n+2}}$

10. Which one of the following expressions denotes the Laplacian of the vector field $\mathbf{F}(x, y, z)$, where $\mathbf{F} = x^3y\mathbf{i} + \ln z\mathbf{j} + \ln(xy)\mathbf{k}$ (9 marks)

(a) $3x^2y\mathbf{i} + \frac{1}{z}\mathbf{j} + 0\mathbf{k}$

(b) $6xy\mathbf{i} - \frac{1}{z^2}\mathbf{j} - \frac{(x^2 + y^2)}{x^2y^2}\mathbf{k}$

(c) $6xy\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$

(d) $3x^2\mathbf{i} - \frac{1}{z^2}\mathbf{j} + 0\mathbf{k}$

11. The function $f(x, y)$ is given by

$$f(x, y) = x^3 + x^2 - xy + y^2 + 4.$$

Calculate, correct to two decimal places, the value of $f(x, y)$ at its saddle point. (9 marks)

12. Find the directional derivative of the scalar field $f(x, y, z) = x^2yz$ at the point with co-ordinates $(1, -1, 1)$ in the direction of the vector $(4, 0, -3)$. (7 marks)