Data provided: Formula Sheet

MAS250

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2020–21**

Mathematics II (Materials)

This is an open book exam.
*Answer **all** questions. The marks awarded to each section of question are shown in italics. The total mark for the paper is 60.*
This exam starts at 10am (GMT), and you must submit your work within two and a half hours (that is, by 12:30pm (GMT)). Late submission will not be considered without extenuating circumstances. Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working in Section B. By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.

Each question in Section A is followed by possible options. Write down the option you believe to be correct (no working needs to be provided for the questions in Section A).

Section A

A1 A card is picked at random from a deck of cards, the suit (clubs, spades, hearts or diamonds) of the card is recorded, and the card is returned to the deck of cards. This process is carried out 100 times, with the following result:

Suit	Clubs	Spades	Hearts	Diamonds
Number of cards	19	32	23	26

Suppose we want to test the hypothesis that a card picked from the deck of cards is equally likely to be of any suit. Suppose π_i is the probability of a suit (for $i = 1, 2, 3, 4$), n is the number of times the process is carried out, and f_i is the number of times suit i is obtained. $\bar{x}$ and s are the mean and standard deviation of the numbers in the table, and $\mu = 100/4 = 25$.

(a) Is the number of degrees of freedom:

(i) 99
(ii) 3
(iii) Something different from the numbers in (i) and (ii) *(1 mark)*

(b) Should we use the statistic:

(i) $\dfrac{\bar{x}-\mu}{s/\sqrt{n}}$

(ii) $\displaystyle\sum_{i=1}^{4} \frac{(f_i - n\pi_i)^2}{n\pi_i}$

(iii) Something different from the statistics in (i) and (ii) *(1 mark)*

(c) Is the value of the appropriate statistic:

(i) 0
(ii) 0.64
(iii) 3.61
(iv) None of (i)-(iii) *(3 marks)*

(d) If we want to test the hypothesis at the 1% level, should we compare the value of the statistic with:

(i) 11.345
(ii) 4.541
(iii) 7.815
(iv) 5.841
(v) 2.364
(vi) None of (i)-(v) *(3 marks)*

 Question A1 continued on next page

A1 (continued)

(e) At the 1% level, would we accept the hypothesis that a card picked from the deck of cards is equally likely to be of any suit:

(i) Yes
(ii) No *(1 mark)*

A2 A vector field $\mathbf{u}$ is given by

$$\mathbf{u} = (x \sin \pi y, x^3 z^2, y^2 z).$$

(a) Is $\nabla(\nabla \cdot \mathbf{u})$ given by:

(i) $(0, -\pi \cos \pi y + 2y, 0)$

(ii) $(0, -\pi \sin \pi y + 2y, 0)$

(iii) $(0, \cos \pi y + 2y, 0)$

(iv) $(0, \pi \cos \pi y + 2y, 0)$

(v) None of (i)-(iv) *(4 marks)*

(b) Is $\nabla^2 \mathbf{u}$ given by:

(i) $(-\pi^2 x \sin \pi y, 6xz^2 + 2x^3, 2z)$

(ii) $(-x \sin \pi y, 6xz^2 + 2x^3, 2z)$

(iii) $(0, 0, 0)$

(iv) $(-\pi^2 x \sin \pi y, 6xz^2, 2z)$

(v) None of (i)-(iv) *(7 marks)*

(c) Is $\nabla \times (\nabla \times \mathbf{u})$ given by:

(i) $(\pi^2 x \sin \pi y, 2x^3 - 2y + 6xz^2 - \pi \cos \pi y, -2z)$

(ii) $(\pi^2 x \cos \pi y, 2y - 2x^3 - 6xz^2 + \pi \cos \pi y, -2z)$

(iii) $(x \sin \pi y, 2y - 2x^3 - 6xz^2 + \cos \pi y, -2z)$

(iv) $(\pi^2 x \sin \pi y, 2y - 2x^3 - 6xz^2 + \pi \cos \pi y, 2z)$

(v) None of (i)-(iv) *(6 marks)*

A3 A partial differential equation for $\phi(x,y)$ is given by

$$\frac{\partial\phi}{\partial y} + x\frac{\partial\phi}{\partial x} + \phi = 0$$

for $x > 0$, $-\infty < y < \infty$.

Suppose $\phi(x,y) = X(x)Y(y)$ for some functions X and Y.

(a) Is X proportional to:

(i) $e^{-\alpha x}$ for some constant α

(ii) $e^{-\alpha/x}$ for some constant α

(iii) $\alpha\left(1+\frac{1}{x}\right)$ for some constant α

(iv) $x^{-\alpha}$ for some constant α

(v) None of (i)-(iv) *(5 marks)*

(b) Is Y proportional to:

(i) e^{xy}

(ii) e^{y}

(iii) $\ln|y|$

(iv) $\ln|\alpha y|$ for some constant α (with $\alpha \neq 1$)

(v) None of (i)-(iv) *(5 marks)*

(c) If $\phi = e^{y}$ when $x = 1$, is ϕ equal to:

(i) $x^{-1}\,e^{y}$

(ii) $x^{-2}\,e^{y}$

(iii) $x^{-2}\,e^{xy}$

(iv) $(2-x)^{-1}\,e^{y}$

(v) None of (i)-(iv) *(4 marks)*

Section B

B1 The function $f(x) = 1 - x^2$ is defined on the interval $0 \leqslant x \leqslant 1$.

(a) Derive the Fourier cosine series for $f(x)$. *(16 marks)*

(b) Use the result of part (a) to find

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$ *(4 marks)*

End of Question Paper

FORMULA SHEET

Trigonometry

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \mathrm{cosec}^2\theta$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

$\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

$a\cos\theta + b\sin\theta = R\cos(\theta - \alpha)$, where $R = \sqrt{(a^2+b^2)}$, $\cos\alpha = a/R$ and $\sin\alpha = b/R$

Hyperbolic Functions

$\sinh x = \dfrac{1}{2}(e^x - e^{-x})$

$\cosh x = \dfrac{1}{2}(e^x + e^{-x})$

$\cosh^2 x - \sinh^2 x = 1$

$\mathrm{sech}^2 x + \tanh^2 x = 1$

$2\sinh x\cosh x = \sinh 2x$

$\cosh 2x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1$

$\sinh^{-1} x = \ln\left[x + \sqrt{(1+x^2)}\right]$, all x

$\cosh^{-1} x = \ln\left[x + \sqrt{(x^2-1)}\right]$, $x \geqslant 1$

$\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$, $\mid x \mid < 1$

$\coth^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right)$, $\mid x \mid > 1$

Differentiation and Integration

Function	Derivative
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
e^x	e^x
$\tan x$	$\sec^2 x$
$\cot x$	$-\text{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\text{cosec} x$	$-\text{cosec} x \cot x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\text{sech}^2 x$
$\coth x$	$-\text{cosech}^2 x$
$\text{sech} x$	$-\text{sech} x \tanh x$
$\text{cosech} x$	$-\text{cosech} x \coth x$
$\sin^{-1} x$	$\frac{1}{\sqrt{(1-x^2)}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{(1-x^2)}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cot^{-1} x$	$-\frac{1}{1+x^2}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{(x^2+1)}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{(x^2-1)}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}, \quad \mid x \mid < 1$
$\coth^{-1} x$	$-\frac{1}{x^2-1}, \quad \mid x \mid > 1$

Function	Integral
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
$\frac{1}{a^2-x^2}$	$\frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\frac{x}{a}\right)$

Differentiation and Integration Formulae

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\int_a^b uv dx = [u \times \text{(integral of } v)]_a^b - \int_a^b \frac{du}{dx} \times \text{(integral of } v) dx$$

Partial Differentiation

Chain Rule

1. Suppose that $z = f(x, y)$ and that x and y are functions of t, i.e., $x = x(t)$, $y = y(t)$. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

2. Suppose that $z = f(x, y)$ and that x and y are functions of the variables r and s, i.e., $x = x(r, s)$, $y = y(r, s)$. Then

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}, \qquad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

First-Order Differential Equations

1. **Direct Integration**

$$\frac{dy}{dx} = f(x)$$

$$y = \int f(x)dx + C$$

2. **Separation of Variables**

$$\frac{dy}{dx} = f(x)g(y)$$

$$\int \frac{dy}{g(y)} = \int f(x)\,dx$$

3. **Homogeneous Equations**

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

make the substitution $y = zx$ to give

$$z + x\frac{dz}{dx} = f(z)$$

4. **Linear Equations**

$$\frac{dy}{dx} + P(x)y = Q(x)$$

multiply both sides by the integrating factor $e^{\int P(x)dx}$ to give

$$\frac{d}{dx}\left(ye^{\int P(x)dx}\right) = Q(x)e^{\int P(x)dx}$$

The Second-Order Differential Equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

where a, b, and c are constants.
General solution is

$$y = \text{Complementary Function} + \text{Particular Integral}$$

The solution, y_c, is given by

(i) $y_c = Ae^{m_1 x} + Be^{m_2 x}$, if m_1 and m_2 real and different,

(ii) $y_c = e^{mx}(A + Bx)$, if m_1 and m_2 real and equal ($m_1 = m_2 = m$),

(iii) $y_c = e^{px}(A\cos qx + B\sin qx)$, if m_1 and m_2 are complex ($m_1 = p + iq, m_2 = p - iq$), where m_1 and m_2 are the roots of the *auxiliary equation*

$$am^2 + bm + c = 0$$

Particular Integral, y_p

$f(x) = Ax^2 + Bx + C$ $\quad$ $y_p = ax^2 + bx + c$

$f(x) = Ae^{kx}$ $\quad$ $y_p = ae^{kx}$
when k is not one of the roots of the auxiliary equation

$f(x) = Ae^{kx}$ $\quad$ $y_p = axe^{kx}$
when k is one of the roots of the auxiliary equation

$f(x) = A\cos mx + B\sin mx$ $\quad$ $y_p = a\cos mx + b\sin mx$
when $\sin mx$ or $\cos mx$ is not part of the complementary function

$f(x) = A\cos mx + B\sin mx$ $\quad$ $y_p = x(a\cos mx + b\sin mx)$
when $\sin mx$ or $\cos mx$ is part of the complementary function

Fourier Series

Suppose that $f(x)$ is defined on the interval $-l \leqslant x \leqslant l$. The Fourier series for $f(x)$ is given by

$$f(x) \;=\; \frac{a_0}{2} \;+\; \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} \;+\; b_n \sin\frac{n\pi x}{l}\right) \;,$$

where

$$a_n = \;\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x \;,\quad n = 0, 1, 2, \ldots \;,$$

$$b_n = \;\frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x \;,\qquad n = 1, 2, \ldots \,.$$

On the interval $0 \leqslant x \leqslant l$ the Fourier cosine series for $f(x)$ is

$$f(x) \;=\; \frac{a_0}{2} \;+\; \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \;,\quad a_n \;=\; \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

and the Fourier sine series is

$$f(x) \;=\; \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \;,\quad b_n \;=\; \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x \;.$$

Vector Calculus

The gradient of the scalar field $\phi(x, y, z)$ is given by

$$\nabla\phi \;=\; \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) \;.$$

The divergence of a vector field $\mathbf{u}(x, y, z) = (u, v, w)$ is given by

$$\nabla\cdot\mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

The curl of a vector field $\mathbf{u}(x, y, z) = (u, v, w)$ is given by

$$\nabla\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

The Laplacian ∇^2 is given by

$$\nabla^2 \;=\; \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Statistics

For data values $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

Means $\bar{x} = \dfrac{1}{n}\sum_{i=1}^{n} x_i$ etc.

Variances $s_x^2 = \dfrac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \dfrac{1}{n}\sum_{i=1}^{n}(x_i^2) - \bar{x}^2$ etc.

s_x is standard deviation

Covariance $\text{cov}(x, y) = \dfrac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \dfrac{1}{n}\sum_{i=1}^{n}(x_i y_i) - \bar{x}\bar{y}$

Correlation coefficient $r = \dfrac{\text{cov}(x, y)}{s_x s_y}$

Linear regression by least squares

The least squares fit to the linear relationship

$$y = a + b(x - \bar{x})$$

is given by

$$a = \bar{y}, \quad b = \frac{\text{cov}(x, y)}{s_x^2}$$

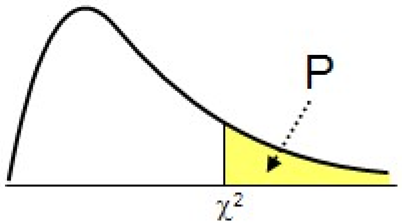
The corresponding mean square residual is $s_y^2(1 - r^2)$.

Table of selected values

The following table lists values for *t*-distributions with ν degrees of freedom for a range of *one-sided* or *two-sided* critical regions. The first column is ν, and the percentages along the top are confidence levels.

One-sided	**75%**	**80%**	**85%**	**90%**	**95%**	**97.5%**	**99%**	**99.5%**	**99.75%**	**99.9%**	**99.95%**
Two-sided	**50%**	**60%**	**70%**	**80%**	**90%**	**95%**	**98%**	**99%**	**99.5%**	**99.8%**	**99.9%**
1	1.000	1.376	1.963	3.078	6.314	12.71	31.82	63.66	127.3	318.3	636.6
2	0.816	1.080	1.386	1.886	2.920	4.303	6.965	9.925	14.09	22.33	31.60
3	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	7.453	10.21	12.92
4	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.610
5	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	4.773	5.893	6.869
6	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	4.317	5.208	5.959
7	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.029	4.785	5.408
8	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.587
11	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	3.497	4.025	4.437
12	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318
13	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.372	3.852	4.221
14	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
15	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073
16	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	3.252	3.686	4.015
17	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	3.222	3.646	3.965
18	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.197	3.610	3.922
19	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883
20	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.850
21	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.135	3.527	3.819
22	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.119	3.505	3.792
23	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.104	3.485	3.767
24	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745
25	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725
26	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707
27	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.690
28	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.047	3.408	3.674
29	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.038	3.396	3.659
30	0.683	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.030	3.385	3.646
40	0.681	0.851	1.050	1.303	1.684	2.021	2.423	2.704	2.971	3.307	3.551
50	0.679	0.849	1.047	1.299	1.676	2.009	2.403	2.678	2.937	3.261	3.496
60	0.679	0.848	1.045	1.296	1.671	2.000	2.390	2.660	2.915	3.232	3.460
80	0.678	0.846	1.043	1.292	1.664	1.990	2.374	2.639	2.887	3.195	3.416

100	0.677	0.845	1.042	1.290	1.660	1.984	2.364	2.626	2.871	3.174	3.390
120	0.677	0.845	1.041	1.289	1.658	1.980	2.358	2.617	2.860	3.160	3.373
∞	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291



Chi-square distribution

	P										
DF	**0.995**	**0.975**	**0.20**	**0.10**	**0.05**	**0.025**	**0.02**	**0.01**	**0.005**	**0.002**	**0.001**
1	0.0000393	0.000982	1.642	2.706	3.841	5.024	5.412	6.635	7.879	9.550	10.828
2	0.0100	0.0506	3.219	4.605	5.991	7.378	7.824	9.210	10.597	12.429	13.816
3	0.0717	0.216	4.642	6.251	7.815	9.348	9.837	11.345	12.838	14.796	16.266
4	0.207	0.484	5.989	7.779	9.488	11.143	11.668	13.277	14.860	16.924	18.467
5	0.412	0.831	7.289	9.236	11.070	12.833	13.388	15.086	16.750	18.907	20.515
6	0.676	1.237	8.558	10.645	12.592	14.449	15.033	16.812	18.548	20.791	22.458
7	0.989	1.690	9.803	12.017	14.067	16.013	16.622	18.475	20.278	22.601	24.322
8	1.344	2.180	11.030	13.362	15.507	17.535	18.168	20.090	21.955	24.352	26.124
9	1.735	2.700	12.242	14.684	16.919	19.023	19.679	21.666	23.589	26.056	27.877
10	2.156	3.247	13.442	15.987	18.307	20.483	21.161	23.209	25.188	27.722	29.588
11	2.603	3.816	14.631	17.275	19.675	21.920	22.618	24.725	26.757	29.354	31.264
12	3.074	4.404	15.812	18.549	21.026	23.337	24.054	26.217	28.300	30.957	32.909
13	3.565	5.009	16.985	19.812	22.362	24.736	25.472	27.688	29.819	32.535	34.528
14	4.075	5.629	18.151	21.064	23.685	26.119	26.873	29.141	31.319	34.091	36.123
15	4.601	6.262	19.311	22.307	24.996	27.488	28.259	30.578	32.801	35.628	37.697
16	5.142	6.908	20.465	23.542	26.296	28.845	29.633	32.000	34.267	37.146	39.252

17	5.697	7.564	21.615	24.769	27.587	30.191	30.995	33.409	35.718	38.648	40.790
18	6.265	8.231	22.760	25.989	28.869	31.526	32.346	34.805	37.156	40.136	42.312
19	6.844	8.907	23.900	27.204	30.144	32.852	33.687	36.191	38.582	41.610	43.820
20	7.434	9.591	25.038	28.412	31.410	34.170	35.020	37.566	39.997	43.072	45.315
21	8.034	10.283	26.171	29.615	32.671	35.479	36.343	38.932	41.401	44.522	46.797
22	8.643	10.982	27.301	30.813	33.924	36.781	37.659	40.289	42.796	45.962	48.268
23	9.260	11.689	28.429	32.007	35.172	38.076	38.968	41.638	44.181	47.391	49.728
24	9.886	12.401	29.553	33.196	36.415	39.364	40.270	42.980	45.559	48.812	51.179
25	10.520	13.120	30.675	34.382	37.652	40.646	41.566	44.314	46.928	50.223	52.620
26	11.160	13.844	31.795	35.563	38.885	41.923	42.856	45.642	48.290	51.627	54.052
27	11.808	14.573	32.912	36.741	40.113	43.195	44.140	46.963	49.645	53.023	55.476
28	12.461	15.308	34.027	37.916	41.337	44.461	45.419	48.278	50.993	54.411	56.892
29	13.121	16.047	35.139	39.087	42.557	45.722	46.693	49.588	52.336	55.792	58.301
30	13.787	16.791	36.250	40.256	43.773	46.979	47.962	50.892	53.672	57.167	59.703
31	14.458	17.539	37.359	41.422	44.985	48.232	49.226	52.191	55.003	58.536	61.098
32	15.134	18.291	38.466	42.585	46.194	49.480	50.487	53.486	56.328	59.899	62.487
33	15.815	19.047	39.572	43.745	47.400	50.725	51.743	54.776	57.648	61.256	63.870
34	16.501	19.806	40.676	44.903	48.602	51.966	52.995	56.061	58.964	62.608	65.247
35	17.192	20.569	41.778	46.059	49.802	53.203	54.244	57.342	60.275	63.955	66.619
36	17.887	21.336	42.879	47.212	50.998	54.437	55.489	58.619	61.581	65.296	67.985
37	18.586	22.106	43.978	48.363	52.192	55.668	56.730	59.893	62.883	66.633	69.346
38	19.289	22.878	45.076	49.513	53.384	56.896	57.969	61.162	64.181	67.966	70.703
39	19.996	23.654	46.173	50.660	54.572	58.120	59.204	62.428	65.476	69.294	72.055
40	20.707	24.433	47.269	51.805	55.758	59.342	60.436	63.691	66.766	70.618	73.402