Data provided: Formula sheet

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2020–21**

Further Civil Engineering Mathematics and Computing

This is an open book exam.
*Answer **all** questions. This exam starts at 10am (GMT), and you must submit your work within two and a half hours (that is, by 12:30pm (GMT)). **Late submission will not be considered without extenuating circumstances.** Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working. By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.*

1 Find and classify all the stationary points of the function

$$z(x, y) = \frac{1}{3}x^3 + \frac{1}{2}y^2 - 2xy - 3x - 3y + 4.$$

(12 marks)

2 The temperature of a metal plate is given by $T(x, y) = x^2e^y - xy^3$ (in arbitrary units), where x and y describe the position on the plate.

(i) A student performs a measurement of the temperature and finds that the temperature at $x = 1$ and $y = 1$ is $T(1, 1) = 1.72$. Next, the student moves to a close position where the value of x has changed to 0.98 and in the new position the temperature is 1.75, however the student cannot determine the change in y. Estimate the change in y using the small increment formula. Give your answer with a *two* decimal place accuracy. ***(5 marks)***

 Question 2 continued on next page

2 (continued)

(ii) The student decides to take measurement along circles of constant radius along which $x = \cos(\theta)$ and $y = \sin(\theta)$, where $0 \le \theta \le 2\pi$ is the angle variable. Use the chain rule to determine the variation of the temperature with the angle θ, i.e. $dT/d\theta$. Find the value of this derivative at $\theta = \pi$.

(8 marks)

3 The oscillations in a homogeneous steel rod with fixed ends are given by the differential equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + 2p \frac{\partial u}{\partial t} = 0, \quad 0 \le x \le 2,$$

where $u = u(x, t)$ is a small transverse displacement of the axis of the rod, c is the propagation speed and $p > 0$ is the damping coefficient. Oscillations are subject to the boundary conditions

$$u(0, t) = u(2, t) = 0,$$

and the initial conditions

$$\frac{\partial u}{\partial t}(x, t = 0) = 0, \quad u(x, 0) = \begin{cases} \dfrac{x}{2}, & \text{for } 0 \le x \le 1, \\ 1 - \dfrac{x}{2}, & \text{for } 1 \le x \le 2. \end{cases}$$

Use the method of separation of variables to find an analytical solution of the above differential equation subject to the imposed boundary and initial conditions.

(25 marks)

4 If air resistance is proportional to the square of the instantaneous velocity, then the velocity $v(t)$ of a mass m dropped from a given height is determined by the differential equation

$$m \frac{dv}{dt} = mg - kv^2$$

where $g = 9.8$ m/s^2 is the gravitational acceleration and $k = 0.125$ kg/m is the air resistance coefficient. Let $v(0) = 0$ and $m = 0.5$ kg. Use the fourth-order Runge-Kutta method with $h = 0.1$ to approximate the value of the velocity at $t = 0.2$. Work correct to *four* decimal places. ***(10 marks)***

End of Question Paper

Formula sheet

- **Trigonometric identities**

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

- **The local truncation error in the case of the 4th order Runge-Kutta method is given by**

$$Y(x) - y(x) = Ch^4$$

where $Y(x)$ is the exact value, $y(x)$ is the estimated numerical value, C is a constant and h is the step size used in the numerical scheme.

- **Chain rule**

If $z = f(x, y)$, where x and y are both functions of t, so that $x = x(t)$ and $y = y(t)$ we have

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

If $z = f(x, y)$ and both x and y are functions of u and v, so that $x = x(u, v)$ and $y = y(u, v)$ then we have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

- **Fourier series**

If the function $f(x)$ is defined over the interval $-l \le x \le l$, then the Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

where

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\ dx, \quad (n = 0, 1, 2, \dots)$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\ dx \quad (n = 1, 2, 3, \dots)$$

If the function $f(x)$ is defined over the interval $0 \leq x \leq l$, then the Fourier cosine series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}, \qquad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\ dx, \quad (n = 0, 1, 2, \dots)$$

while the sine series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \qquad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\ dx \quad (n = 1, 2, 3, \dots)$$

- **the orthogonality of the sine function can be defined as**

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & \textbf{if } m \neq n \\ L/2 & \textbf{if } m = n \end{cases}$$