The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS Autumn Semester 2020–2021

Continuum Mechanics

This is an open book exam.
*Answer **all** questions.*
You can work on the exam during the 24 hour period starting at 10am (GMT), and you must submit your work within 3.5 hours of accessing the exam paper or by the end of the 24 hour period (whichever is earlier)." Late submission will not be considered without extenuating circumstances. *Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working. By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.*

1 (i) Write down in full the following expressions:

$$\text{(a)}\ \delta_{ij}U_{ij}, \quad \text{(b)}\ t_i = \varepsilon_{ijk}U_{jk}, \quad \text{(c)}\ v_i = T_{ij}u_j.$$

(8 marks)

 Question 1 continued on next page

1 (continued)

(ii) The transformation matrix from old to new Cartesian coordinates, $\hat{\mathbf{A}}$, is

$$\hat{\mathbf{A}} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrix of tensor $\mathbf{T}$ in the old coordinates is given by

$$\hat{\mathbf{T}} = \begin{pmatrix} 2 & a & 0 \\ b & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

(a) Use the relation $\hat{\mathbf{T}}' = \hat{\mathbf{A}}\hat{\mathbf{T}}\hat{\mathbf{A}}^T$ between the matrices of tensor $\mathbf{T}$ in the old and new coordinates to calculate the matrix of tensor $\mathbf{T}$ in the new coordinates. ***(7 marks)***

(b) You are given that the matrix of tensor $\mathbf{T}$ in the new coordinates is diagonal, $T'_{ij} = 0$ for $i \neq j$. Determine a and b. ***(4 marks)***

(iii) At the initial time, $t = 0$, a continuum occupies a cylinder of infinite length and radius R. It starts to expand preserving its cylindrical shape. The velocity of any point at the surface of the cylinder is perpendicular to the surface and its magnitude is equal to $a^2V/(a^2 + t^2)$, where a is a constant with the dimension of time. Inside the cylinder the velocity is also in the radial direction and its magnitude is proportional to the distance from the centre of the cylinder.

(a) Find the radius of the cylinder at time t. ***(4 marks)***

(b) Find the dependence of the velocity v on the distance from the centre of the cylinder, r, at time t. ***(3 marks)***

(c) You are given that the density in the cylinder has the form $\rho = rf(t)$, and $f(0) = \rho_0/R$. Use the mass conservation equation for axisymmetric motion in cylindrical coordinates to determine $f(t)$. Calculate the limiting value of the density as $t \to \infty$. ***(12 marks)***

 Question 1 continued on next page

1 (continued)

(iv) A continuous medium occupies the exterior of a solid sphere of radius R. The sphere is charged and the medium is dielectric. The electrical field produced by the sphere creates the polarisation electrical charge in the medium. As a result, there is the body force in the medium equal to $\boldsymbol{b} = -E(R/r)^4\boldsymbol{e}_r$, where $\boldsymbol{e}_r$ in the unit vector in the radial direction in spherical coordinates r, θ, φ with the origin at the centre of the sphere, and $E > 0$ is a constant. You are also given that the stress tensor in the medium has the form $\mathbf{T} = -p\mathbf{I}$, where $\mathbf{I}$ is the unit tensor, p is the pressure related to the density ρ by $p = a\rho$ with $a > 0$, and $p = p_0 =$ const at $r = R$. Use the equilibrium equation

$$\nabla p = \rho \boldsymbol{b},$$

to determine the dependence of p on r for $r > R$. Calculate the limiting value of pressure, p_∞, as $r \to \infty$. ***(12 marks)***

2 (i) (a) Use Bernoulli's integral for an ideal incompressible fluid to show that the water pressure is given by

$$p = p_a - \rho g z,$$

where p is the water pressure, p_a the atmospheric pressure, ρ the water density, g the gravitational acceleration, and the z-axis is directed upwards with $z = 0$ at the water surface. ***(6 marks)***

(b) Prove Archimedes' law: the pressure force exerted on the surface of a body immersed in water is in the vertical direction, and its magnitude is equal to the weight of water displaced by the body. ***(9 marks)***

(c) A bathysphere has the shape of a sphere of radius $r = 1.5$ m. It is completely immersed in water and attached to a ship by a steel rope. The bathysphere mass is 16×10^3 kg. What is the tension T in the rope? (You can take the water density $\rho = 10^3$ kg/m^3 and $g = 10$ m/s^2.) ***(6 marks)***

 Question 2 continued on next page

2 (continued)

(ii) (a) You are given that, in equilibrium, the stress tensor $\mathbf{T}$ satisfies the equation written in Cartesian coordinates x_1, x_2, x_3,

$$\frac{\partial T_{ij}}{\partial x_j} + \rho b_i = 0, \qquad (*)$$

where T_{ij} are the components of the stress tensor $\mathbf{T}$, ρ is the density, and b_i are the components of the body force $\boldsymbol{b}$. You are also given that, in linear elasticity, the Cartesian components of the stress tensor are given by

$$T_{ij} = \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \qquad (\dagger)$$

where u_i are the Cartesian components of the displacement $\boldsymbol{u}$, and λ and μ are the Lamé constants. Show that, when $b_i = 0$, in the linear elasticity equation $(*)$ reduces to

$$(\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) + \mu \nabla^2 \boldsymbol{u} = 0. \qquad (\ddagger)$$

(10 marks)

(b) There is a slab of elastic material. The slab thickness if $h = 10$ cm and it is infinite in two other directions, so, in Cartesian coordinates x, y, z the elastic material occupies the region $0 \le z \le h$. There are forces applied to the slab boundaries. The density of each of the two forces is equal to f. The force applied at the upper boundary ($z = h$) is in the positive x-direction, while the force applied at the lower boundary ($z = 0$) is in the negative x-direction. You are given that the displacement both at $z = 0$ and $z = h$ is equal to $U = 2 \times 10^{-4}$ m, and it is in the positive x-direction at $z = h$ and in the negative x-direction at $z = 0$. You are also given that $\lambda = \mu = 8 \times 10^{10}$ N m^{-2}, where λ and μ are the Lamé constants. Use the equation of the linear elasticity $(\ddagger)$ and the boundary conditions at the slab boundaries to find f.

Hint: Assume that the displacement $\boldsymbol{u}$ has only the x-component that depends only on z. ***(19 marks)***

End of Question Paper