The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2020–21**

Introduction to Relativity

This is an open book exam.
Answer all questions.
You can work on the exam during the 24 hour period starting at 10am (GMT), and you must submit your work within 2.5 hours (150 minutes) of accessing the exam paper or by the end of the 24 hour period, whichever is earlier. ***Late submission will not be considered without extenuating circumstances.*** *Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working. By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.*

1 An astronaut A is at rest at the origin of an inertial frame R.
A spaceship S passes A, moving at instantaneous velocity v in the positive-x direction of R.
At the moment of passing, the astronaut's clock shows $t = 0$ and the spaceship's clock shows $\tau = 0$.

(i) (a) Define *inertial frame*.
Define the *rapidity* ρ of the spaceship S in R.
Write down the four-velocity V of the spaceship in R in terms of its rapidity. ***(4 marks)***

(b) At rest, the spaceship is spherical with radius r. Describe the shape of the moving spaceship as seen by the astronaut A. ***(2 marks)***

(ii) For $t > 0$, the spaceship S accelerates in the $-x$ direction of R with a constant acceleration of magnitude a in its instantaneous rest frame.
The spaceship's rapidity ρ in R is a function of the proper time τ on the spaceship's clock.
When the spaceship passes A for the second time, it is moving at an instantaneous velocity $-v$ in the x-direction in R, and the astronaut's clock shows t_1, and the spaceship's clock shows τ_1.

(a) By calculating the four-acceleration, or otherwise, show that the rapidity is

$$\rho = -\frac{a\tau}{c} + \rho_0,$$

where $\rho_0 = \tanh^{-1}(v/c)$ is a constant. ***(6 marks)***

(b) Calculate the worldline $X(\tau)$ of the spaceship in R.
Sketch the worldlines of A and S on a spacetime diagram. ***(7 marks)***

(c) Find the ratio of t_1/τ_1 in terms of ρ_0. Which time is greater? ***(6 marks)***

2 Photons are massless particles. The four-momentum of a photon of energy $E > 0$ travelling in direction $\mathbf{u}$ in an inertial frame is

$$P_\nu = \frac{E}{c}(1, \mathbf{u}), \qquad \mathbf{u} \cdot \mathbf{u} = 1.$$

(i) Define the Lorentz bracket.
Show that $g(P_\nu, P_\nu) = 0$.
Write down the four-momentum P_m of a particle of mass m.
Derive the *dispersion relation* for a particle of mass m. ***(8 marks)***

(ii) A particle of rest mass m is initially at rest in an inertial frame R. Show that after absorbing a photon of energy E, the particle has a rest mass

$$M = m\sqrt{1 + \frac{2E}{mc^2}}.$$

(5 marks)

(iii) A photon is moving in the $-x$ direction of an inertial frame R, when it is reflected by a perfect mirror that is held stationary in R. The photon's energy E is unchanged by the reflection. The photon's direction in R is reversed. A second inertial frame $\widetilde{R}$ is moving in the $+x$ direction at speed v. By applying a standard Lorentz transformation, or otherwise, find the energy of the photon in $\widetilde{R}$ *before* and *after* the reflection. ***(6 marks)***

(iv) A photon of initial energy E_0 bounces between a pair of parallel perfect mirrors, in deep space. The mirrors are initially at rest in R, and each has a rest mass m such that $mc^2 \gg E_0$. Without detailed calculations, give a *qualitative* account of how this system evolves in time, considering the motion of the mirrors, the energy and momentum of the photon, and the final speed of the mirrors. ***(6 marks)***

End of Question Paper