The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2020–21**

WAVES

This is an open book exam.
Answer ***all*** *questions.*
You can work on the exam during the 24 hour period starting at 10am (GMT), and you must submit your work within 2 hrs and 30 min of accessing the exam paper or by the end of the 24 hour period (whichever is earlier). ***Late submission will not be considered without extenuating circumstances.*** *Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working. You will not get full marks if you simply write down output from a computer package.*
By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.

1 Use D'Alembert's solution to find the solution $u(x,t)$ to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

subject to the initial conditions

$$u(x,0) = \frac{\sin(x)}{x}, \qquad \frac{\partial u}{\partial t}(x,0) = \frac{1}{1+x^2}.$$

(10 marks)

2 Calculate the Fourier series of the periodic function $f(t)$ with fundamental period $T = 4$ defined on $[-2, 2)$ by

$$f(t) \;=\; \begin{cases} 1 - |t| & -1 \le t \le 1; \\ 0 & \text{otherwise.} \end{cases}$$

(10 marks)

3 Consider standing waves on a uniform rectangular elastic membrane with sides of lengths a, b and with uniform tension. The origin O of a Cartesian coordinate system $(x,\ y,\ z)$ is taken at one corner of the membrane with the z-axis perpendicular to the membrane. After a small perturbation the transversal deflection $z(x, y, t)$ of the membrane from its equilibrium at position (x, y) and time t satisfies the two-dimensional wave equation

$$z_{tt} = c^2(z_{xx} + z_{yy}),$$

where the constant c has the units of velocity. The membrane is fixed at its edges therefore the boundary conditions on the deflection are:

$$\begin{aligned} z(0, y, t) = z(a, y, t) = 0, &\quad 0 \le y \le b, t \ge 0, \\ z(x, 0, t) = z(x, b, t) = 0, &\quad 0 \le x \le a, t \ge 0. \end{aligned}$$

Using the method of separation of variables *derive* that for any pair $m, n \ge 1$, where $m, n \in N^+$, the z_{mn} normal mode is

$$z_{mn}(x, y, t) = \sin \alpha_m x \cdot \sin \beta_n y \cdot (A_{mn} \cos \lambda_{mn} t + B_{mn} \sin \lambda_{mn} t),$$

where α_m, β_n are wave numbers in the $x-, y-$directions, respectively, and λ_{mn} is frequency. Determine λ_{mn} in terms of α_m, β_n and state it when $a = b$.

(10 marks)

4 The equilibrium position of the free surface of an inviscid incompressible and irrotational shallow liquid of depth h is $z = 0$, where z is measured vertically upwards. A linear surface wave causes the displacement of this surface to be $\eta(x, t)$, where x is measured along the free surface and

$$\eta = a \sin kx \sin \omega t,$$

where a, k and ω are positive constants. You are given that the velocity potential

$$\phi = f(z) \sin kx \cos \omega t$$

for a suitable $f(z)$, satisfies the relevant boundary conditions and the equation modelling the incompressible and irrotational nature of the liquid. Use this modelling equation and the appropriate boundary conditions to (i) find $f(z)$ and (ii) derive the accompanying dispersion relation $\omega(k)$.

(10 marks)

5 Using the method of characteristics solve the equation

$$xz_x - 2yz_y = z^2,$$

given that $z = x^3$ on $x = y$. $\left[\textit{We use the notation } z_x = \dfrac{\partial z}{\partial x} \textit{ etc.} \right]$

(10 marks)

End of Question Paper