The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2020–21**

Complex Analysis

This is an open book exam.
Answer all questions.
You can work on the exam during the 24 hour period starting at 10am (GMT), and you must submit your work within 3 hours of accessing the exam paper or by the end of the 24 hour period (whichever is earlier). ***Late submission will not be considered without extenuating circumstances.*** *Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working. By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.*

1 The first two parts of this question use the following four contour integrals. In each of them, C is the unit circle $|z| = 1$ in the anti-clockwise orientation.

$$\int_C \frac{\cos(z)^3}{\sin(z^5)} \mathrm{d}z \tag{1}$$

$$\int_C \frac{\sin(z)}{2z^2 - 5z + 2} \mathrm{d}z \tag{2}$$

$$\int_C |z-2|^2 \mathrm{d}z \tag{3}$$

$$\int_C \frac{\cos(z)}{2z^2 + z + 6} \mathrm{d}z \tag{4}$$

(i) State which one of the four integrals above is well suited to be evaluated using Cauchy's Integral Formula, and then use Cauchy's Integral Formula to evaluate it. Finally, explain why each of the other three integrals *aren't* good choices for applying Cauchy's Integral Formula. ***(6 marks)***

(ii) Using any methods from the module that you would like, evaluate the remaining three integrals, being sure to justify your work. ***(13 marks)***

(iii) Suppose that $f(z)$ is analytic on $\mathbb{C}$, and let $u(x, y)$ and $v(x, y)$ be the real and imaginary parts of f. In other words, for x, y in $\mathbb{R}$, $f(x+iy) = u(x, y) + iv(x, y)$.

Prove that if $u(x, y) + x = v(x, y)$, then

$$f(z) = \frac{i-1}{2} z + (1+i)c$$

for some constant $c \in \mathbb{R}$. ***(6 marks)***

2 Define $f(z) = \dfrac{1-e^{iz}}{z^2}$, and let A and B be the integrals of the real and imaginary parts of f along the real line, namely:

$$A = \int_{-\infty}^{\infty} \frac{1-\cos(x)}{x^2}\mathrm{d}x \qquad B = \int_{-\infty}^{\infty} \frac{-\sin(x)}{x^2}\mathrm{d}x$$

We will also make use of the following paths, where r and R are real constants with $0 < r < R$:

$$\begin{array}{ll} \gamma_r(t) = re^{\pi(1-t)i} & 0 \leq t \leq 1 \\ \alpha_{r,R}(t) = t & -R \leq t \leq -r \\ \beta_{r,R}(t) = t & r \leq t \leq R \\ \gamma_R(t) = Re^{\pi t i} & 0 \leq t \leq 1 \end{array}$$

(i) Using Taylor series, explain why integral A converges at 0, but integral B diverges at 0.

Further explain why we can't use Theorem 12.1 from the lecture notes to evaluate integral A. ***(4 marks)***

(ii) Sketch the contour $\Gamma_{r,R} = \alpha_{r,R} + \gamma_r + \beta_{r,R} + \gamma_R$, labelling each of the four paths. Then prove that $\int_{\Gamma_{r,R}} f(z)\mathrm{d}z = 0$.

(4 marks)

(iii) Prove that for z on γ_R, $|f(z)| \leq 2/R^2$. Deduce that

$$\lim_{R\to\infty} \int_{\gamma_R} f(z)\mathrm{d}z = 0$$

(4 marks)

(iv) By considering Laurent series, show that $f(z) = -i/z + g(z)$ for an analytic function $g(z)$ with $g(0) = 1/2$. Further show that:

$$\int_{\gamma_r} -i/z\mathrm{d}z = -\pi \quad \text{and} \quad \lim_{r\to 0} \int_{\gamma_r} g(z)\mathrm{d}z = 0$$

and hence deduce that

$$\lim_{r\to 0} \int_{\gamma_r} f(z)\mathrm{d}z = -\pi$$

(6 marks)

(v) By looking at the real part of the integral in Part (ii) and taking an appropriate limit, prove that $A = \pi$. You may use the results of the previous parts even if you couldn't prove them. ***(4 marks)***

(vi) Although we know from Part (i) that B diverges, your friend Ashley proposes that arguing as in Parts (ii)-(v) but taking the imaginary part in Part (v) would give a finite value for B. Explain where Ashley's argument breaks down. ***(3 marks)***

End of Question Paper