The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2020-21**

Fields

This is an open book exam.
*Answer **all** questions.*
*You can work on the exam during the 24 hour period starting at 10am (GMT), and you must submit your work within 3 hours of accessing the exam paper or by the end of the 24 hour period." **Late submission will not be considered without extenuating circumstances.** Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working. By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.*

1 Let m and n be positive integers, $a = 5^{\frac{1}{n}}$ and $b = 5^{\frac{1}{nm}}$, $K = \mathbb{Q}(a)$ and $L = \mathbb{Q}(b)$.

(i) Show that $K \subseteq L$ and $[L : K] = m$. ***(12 marks)***

(ii) Find the minimal polynomial $m(x)$ of the element b over the field K. ***(5 marks)***

(iii) Let $c = 5^{\frac{1}{m}}$ and $M = \mathbb{Q}(a, c)$. Show that $K \subseteq M \subseteq L$ and find the degree $[L : M]$. (Hint: Let l and g be the least common multiple and the highest common factor of the numbers n and m, respectively. You may assume that $nm = gl$). ***(18 marks)***

2 (i) Which of the following regular n-gons can be constructed:

$$n = 40, \ \ n = 100 \ \text{ and } \ n = 340.$$

Justify your response. ***(7 marks)***

(ii) Let $a \in \mathbb{R}$ be a constructible number. Show that every element b of the field $\mathbb{Q}(a)$ is also a constructible number. ***(8 marks)***

3 Let K, L and M be subfields of a field F such that $K \subseteq L$, $K \subseteq M$, $[L : K] = n < \infty$ and $[M : K] = m < \infty$. Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$ be K-bases of the fields L and M over K, respectively. Let LM be the set of all finite sums $l_1 m_1 + \cdots + l_s m_s$ where $l_i \in L$ and $m_i \in M$.

(i) Prove that the set LM is a field. ***(8 marks)***

(ii) Prove that $[LM : K] \leq nm$. ***(5 marks)***

(iii) Prove that $[LM : K] = nm$ if and only if the set $C = \{a_i b_j \mid i = 1, \ldots, n; j = 1, \ldots, m\}$ is a K-basis for LM. ***(7 marks)***

End of Question Paper