The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2020–21**

Differential Geometry

This is an open book exam.
Answer all questions.
You can work on the exam during the 24 hour period starting at 10am (GMT), and you must submit your work within three hours of accessing the exam paper or by the end of the 24 hour period (whichever is earlier). ***Late submission will not be considered without extenuating circumstances.*** *Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working. By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.*

A list of formulae is provided on the last two pages.

1 (i) The parametrized curve $\gamma_1 \colon \mathbb{R} \to \mathbb{R}^2$, $\gamma_1(t) := \left(\sinh^{-1}(t), \sqrt{1+t^2}\right)$ has curvature function $t \mapsto 1/(1+t^2)$. Use this to find a curve $\overline{\gamma}_1 \colon \mathbb{R} \to \mathbb{R}^2$ which has curvature function $t \mapsto 1/(1+t^2)$, with $\overline{\gamma}_1(0) = (-1, 0)$ and $\dot{\overline{\gamma}}_1(0) = (0, -1)$. Justify your reasoning. ***(6 marks)***

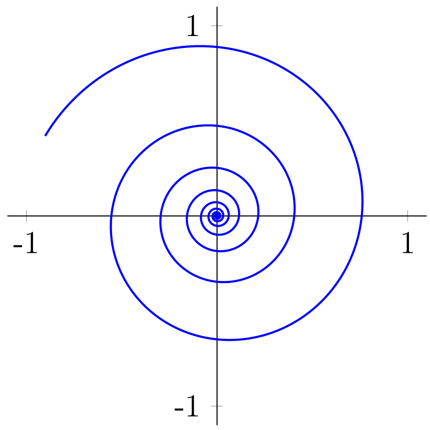
(ii) State whether each of the following assertions is true or false and give *careful* justification of your answer. Results used from the course should be precisely stated.

(a) For all $n \in \{1, 2, 3, \dots\}$, the parametrized curve $\tilde{\gamma}_2 \colon \mathbb{R} \to \mathbb{R}^2$, $\tilde{\gamma}_2(t) := (t^n, t^n)$ is a reparametrization of the curve $\gamma_2 \colon \mathbb{R} \to \mathbb{R}^2$, $\gamma_2(t) := (t, t)$.

(b) There is a unit-speed parametrization $\gamma_3 \colon \mathbb{R} \to \mathbb{R}^2$, of the level set $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 4\}$ which has a turning angle $\theta \colon \mathbb{R} \to \mathbb{R}$ given by $\theta(t) := t + \pi/2$.

(11 marks)

(iii) The following is the image of a unit-speed parametrized curve $\gamma \colon\ ]0, 10[\to \mathbb{R}^2$, which has $\lim_{t \to 0} \gamma(t) = (0, 0)$. Sketch the graph of the curvature function and indicate clearly all relevant features of your sketch.



(8 marks)

2 (i) Consider the parametrized surface $\sigma\colon \mathbb{R}\times]0,\infty[\to \mathbb{R}^3$ given by

$$\sigma(u,v) = (x(u,v), y(u,v), z(u,v)) := \frac{1}{2\sqrt{u^2+v^2}}\big(u^2-v^2, 2uv, \sqrt{3}(u^2+v^2)\big).$$

The matrix representing its first fundamental form is $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ for every point in the domain.

(a) By computing $x(u,v)^2 + y(u,v)^2$ and comparing it to $z(u,v)$, or otherwise, find an equation satisfied by the points in the image of the parametrized surface σ. Either sketch the set of solutions of that equation, or describe its shape in words. If the domain represents a piece of paper what does the first fundamental form suggest is, or is not, happening to the piece of paper; justify your answer.

(6 marks)

(b) Give a curve $\lambda\colon [\alpha,\beta] \to \mathbb{R}\times]0,\infty[\subset \mathbb{R}^2$, for some $\alpha, \beta \in \mathbb{R}$, such that the curve on the surface $\sigma\circ\lambda\colon [\alpha,\beta] \to \mathbb{R}^3$ joins the points $\frac{1}{\sqrt{2}}(0,1,\sqrt{3})$ and $\frac{1}{\sqrt{2}}(0,-1,\sqrt{3})$. Find the length of that curve on the surface. Justify your reasoning. (Contrary to our usual convention, the domain of the curve is a closed interval.)

(5 marks)

(ii) Consider the parametrized surface $\rho\colon \mathbb{R}\times]0,\infty[\to \mathbb{R}^3$ given by

$$\rho(u,v) = \big(\cos u - v\sin u, \sin u + v\cos u, 2(v+u)\big).$$

This has Weingarten matrix $W_{(u,v)} = \begin{pmatrix} -\frac{2}{\sqrt{5}v} & 0\\ \frac{2}{\sqrt{5}v} & 0\end{pmatrix}$.

(a) Find the principal curvatures and the corresponding normalized principal vectors. ***(9 marks)***

(b) Show that every point $\rho(u_0, v_0)$ on the surface is contained in a straight line in the surface which goes in the v-direction.

(c) Explain what happens to the preferred unit normal vector to the surface as you move in the v-direction at each point.

(d) Explain why your answer to (c) does or does not follow from your answer to (b). ***(5 marks)***

End of Question Paper

List of formulae

- The inverse of a 2×2-matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with coefficients in $\mathbb{R}$ and $ad-bc\neq 0$ is
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$
- The cross-product of two vectors $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$ is
$$v_1 \times v_2 = (y_1z_2 - z_1y_2, z_1x_2 - z_2x_1, x_1y_2 - x_2y_1) \in \mathbb{R}^3.$$
- The angle θ between two vectors v_1 and $v_2 \in \mathbb{R}^3$ is given by
$$\cos\theta = \frac{v_1 \cdot v_2}{\|v_1\|\,\|v_2\|}.$$

Here are some hyperbolic identities.

- $\cosh^2(x) + \sinh^2(x) = \cosh(2x)$
- $\cosh^2(x) - \sinh^2(x) = 1$

Inverse hyperbolic functions are given by the following.

- $\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$
- $\cosh^{-1}(x) = \ln(x + \sqrt{x^2-1})$ for $x \geq 1$
- $\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ for $|x| < 1$

Derivatives of some functions are given by the following.

- $\frac{d}{dx}\operatorname{sech}(x) = -\tanh(x)\operatorname{sech}(x)$
- $\frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x)$
- $\frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{x^2+1}}$
- $\frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}}$ for $x > 1$
- $\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}$ for $|x| < 1$
- $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$

Some relations between trigonometric functions are the following.

- $\sin(\tan^{-1}x) = \frac{x}{\sqrt{1+x^2}}$
- $\cos(\tan^{-1}x) = \frac{1}{\sqrt{1+x^2}}$
- $\cos(\theta) = \sin(\theta + \pi/2)$

For a parametrized curve $\gamma\colon\,]\alpha,\beta[\, \to \mathbb{R}^2$, $\gamma(t) = (x(t), y(t))$ we have the following.

- The arc length from $\gamma(a)$ to $\gamma(b)$, $\alpha < a \leq b < \beta$ is
$$\int_a^b \|\dot\gamma(t)\|\, dt.$$
- The curvature of γ at t is
$$\kappa(t) = \frac{\ddot\gamma(t)\cdot J(\dot\gamma(t))}{\|\dot\gamma(t)\|^3} = \frac{x'(t)y''(t) - y'(t)x''(t)}{[x'(t)^2 + y'(t)^2]^{3/2}},$$
where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the anti-clockwise rotation of angle $\pi/2$.

For a parametrized surface $\sigma\colon U \to \mathbb{R}^3$, with U an open set in $\mathbb{R}^2$ the following hold.

- The matrix of the first fundamental form is given by
$$\mathrm{I}_{(u,v)} = \begin{pmatrix} E(u,v) & F(u,v) \\ F(u,v) & G(u,v) \end{pmatrix}$$
for all $(u,v) \in \mathbb{R}^2$, with $E = \sigma_u \cdot \sigma_u$, $F = \sigma_u \cdot \sigma_v$ and $G = \sigma_v \cdot \sigma_v$.
- The area of the domain $\sigma\left([\alpha_1,\beta_1]\times[\alpha_2,\beta_2]\right)$, for $[\alpha_1,\beta_1]\times[\alpha_2,\beta_2] \subseteq U$ is given by
$$\int_{u=\alpha_1}^{\beta_1}\int_{v=\alpha_2}^{\beta_2} \sqrt{EG - F^2}\, dv\, du$$
- The preferred unit normal vector along σ is given by $\mathbf{n}\colon U \to \mathbb{R}^3$,
$$\mathbf{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}.$$
- The matrix of the second fundamental form of σ at $(u,v) \in U$ is
$$\mathrm{II}_{(u,v)} = \begin{pmatrix} L(u,v) & M(u,v) \\ M(u,v) & N(u,v) \end{pmatrix}$$
where $L = \sigma_{uu}\cdot\mathbf{n}$, $M = \sigma_{uv}\cdot\mathbf{n}$ and $N = \sigma_{vv}\cdot\mathbf{n}$.
- The Weingarten matrix of σ is
$$W = \mathrm{I}^{-1}\,\mathrm{II}\,.$$
- The Gaussian curvature is
$$K = \det W.$$

The Brioschi formula:
$$K = \frac{\begin{vmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix}}{(EG - F^2)^2}.$$