The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2020–21**

Financial Mathematics **3 hours**

This is an open book exam.
Answer both questions. The total marks for this exam is 80.

You can work on the exam during the 24 hour period starting at 10am (GMT), and you must submit your work within 3 hours of accessing the exam paper or by the end of the 24 hour period (whichever is earlier). ***Late submission will not be considered without extenuating circumstances.*** *Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working. By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.*

1 (i) The price of a stock which pays no dividends is currently £8. Over each of the next two 1-year periods the stock price will either increase by 50% or decrease by 50%. Suppose that all interest rates are constant and equal to 3%.

(a) Use a binomial tree to find the price of a two-year American put option on this stock with strike price £9. ***(7 marks)***

(b) Describe all circumstances when a rational investor should exercise the option described in (a) before its expiration. ***(3 marks)***

(ii) On October 14th I short-sold, through my broker, 100 bars of copper for \$27.75 per bar. On October 16th, the price was down to \$27.58, so I bought back 100 bars, returned them to their rightful owner, and made myself a small profit of $100 \times (27.75 - 27.58) = \$17$. Was my profit the result of an arbitrage opportunity? Justify your answer. ***(5 marks)***

(iii) Assume that the interest rate is 5%, and consider call and put options of both American and European style expiring in 6 months on non-dividend paying stock. For each of the following scenarios check if you can find an arbitrage opportunity and, if you can, describe it:

(a) The shares are traded at £3 and the American call option is traded at £3.20. ***(5 marks)***

(b) The shares are traded at £3 and a European call option with strike price of £2 is traded at 50p. ***(5 marks)***

(iv) The owner of a bond-like contract, let's call it an Abond, is entitled to three payments of £p_i at times $t_i, i = 1, 2, 3$ as well as a final payment of £X on the maturity date of the Abond t_3 years from now.

(a) Find the spot price of the Abond. ***(7 marks)***

(b) The holder wants to sell her Abond just after receiving the second coupon payment at time t_2, and has entered a short position in a forward contract with a buyer. Show that the forward price L that she agrees for the sale of the Abond is

$$L = (X + p_3)e^{r_2 t_2 - r_3 t_3},$$

where $r_{t_i} = Y(t_i)$, the spot interest rates at t_i. ***(8 marks)***

For this problem you can assume that the yield curve $Y(t)$ is known for any time t, that there is no risk of default, and that there are no arbitrage opportunities in the market.

2 (i) (a) Let S denote the price of a stock paying no dividends, and consider a European call option on this stock with strike price X and expiring in T years. Denote the price of this option by $c(S,t)$. Assume all interest rates are constant and equal to r.

(α) Show that the function

$$f(S,t) = S - Xe^{-r(T-t)}$$

satisfies the Black-Scholes partial differential equation.

(5 marks)

(β) Explain why the result in (a) implies that

$$c(S,t) \geq S - Xe^{-r(T-t)}.$$

(5 marks)

(b) Let $u(x,t)$ be a differentiable function satisfying the PDE $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial u}{\partial t} = 0$. Show that the process $Y_t = u(B_t, t)$ satisfies

$$dY = \frac{\partial u}{\partial x}(B,t)dB.$$

(5 marks)

(ii) You are given the following data on three stocks and the market portfolio:

	Expected return	Correlation with market portfolio	Standard deviation of return
Stock 1	?	0.7	20%
Stock 2	14%	?	25%
Stock 3	4%	-0.5	?
Market portfolio	10%	1	10%

The risk-free interest rate for the period is 5%. Give the equation of the capital market line, find the beta-coefficients of Stocks 1, 2 and 3, and fill in all missing data in the table above. ***(15 marks)***

 Question 2 continued on next page

2 (continued)

(iii) Let A and B be two assets, and let A_t, B_t denote the prices of these t years hence. We say that A *dominates* B if we know with certainty that for some $T > 0$ years

(i) $A_T \geq B_T$, and

(ii) the probability of $A_T > B_T$ is non-zero.

We have seen that if (i) holds with certainty (and no assumption like (ii) is made), then $A_t \geq B_t$ for all $0 \leq t \leq T$. In particular, if $A_T \geq B_T$ with probability 1, then $A_0 \geq B_0$.

In this question we make the following stronger assumption: if A dominates B, i.e., both (i) and (ii) from above hold, then $A_0 > B_0$.

Let p_t, P_t, c_t denote the prices t years hence of a European put option, an American put option and a European call option, respectively, on the same underlying asset providing no income, with same strike price X and all expiring in T years. Let S_t be the price of the underlying asset t years hence. Assume all interest rates are equal to $r > 0$. Let $0 < t < T$.

(a) Use put-call parity to show that $p_t = (X - S_t) + c_t - X(1 - e^{-r(T-t)})$. ***(3 marks)***

(b) Show that $P_t \geq \max\{X - S_t, 0\}$ and use that to show: if $c_t < X(1 - e^{-r(T-t)})$, then $p_t < P_t$. ***(5 marks)***

(c) Deduce that if $c_t < X(1 - e^{-r(T-t)})$ occurs with non-zero probability, then the American put option dominates the European put option and hence $p_0 < P_0$. ***(2 marks)***

End of Question Paper