**Data provided: Formula sheet**

**MAS381**

The University Of Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS** **Autumn Semester 2020–21**

**Mathematics III**

*This is an open book exam.*
*Answer **all** questions. This exam starts at 10am (GMT), and you must submit your work within two and a half hours (that is, by 12:30pm (GMT)). **Late submission will not be considered without extenuating circumstances.** Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working. By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.*

**1** Prove that the function $f(z) = e^{iz^2}$ satisfies the Cauchy-Riemann equations.

***(10 marks)***

**2** Use the Residue Theorem to find the value of the real integral

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+2x+2)(x^2+1)^2}$$

***(20 marks)***

**3** Verify Stokes' theorem for the vector field

$$\mathbf{F} = y^2e^x\mathbf{i} + (x^2 - z)e^y\mathbf{j} + 2ze^{xy}\mathbf{k}$$

defined over the rectangle $0 \le x \le 2$, $0 \le y \le 1$ and $z = 0$. ***(20 marks)***

**4** Find the Laurent expansion of the function

$$f(z) = \frac{z - 1}{3z^2 + 5iz + 2}$$

valid in the interval $1/3 < |z| < 1/2$. ***(10 marks)***

## End of Question Paper

**Formula sheet**

- The general formula for the residue at a pole $z_0$, of order $m$ is

$$\frac{1}{(m-1)!}\lim_{z\to z_0}\left\{\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\right\}$$

- Useful identities

$$\sin 2\theta = 2\sin\theta\cos\theta, \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

- The are element in the $xy$ plane in Cartesian coordinates is given by

$$dA = dxdy, \text{ and } d\mathbf{A} = dA\mathbf{k}$$

- The spherical area and volume elements are given by

$$dS = r^2\sin\theta\ d\theta d\phi, \quad dV = r^2\sin\phi\ dr\ d\phi\ d\theta$$

- The position vector in spherical coordinates for a unit radius is given as

$$\mathbf{r} = (x, y, z) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

- The coordinates between Cartesian and spherical coordinate systems transform according to

$$x = r\sin\phi\cos\theta$$
$$y = r\sin\phi\sin\theta$$
$$z = r\cos\phi$$

- The unit normal vector to the spherical surface is

$$\hat{\mathbf{n}} = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$