The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2020–21**

Topics in Advanced Fluid Mechanics

This is an open book exam.
Answer all questions.
You can work on the exam during the 24 hour period starting at 10am (GMT), and you must submit your work within 3 hours of accessing the exam paper or by the end of the 24 hour period (whichever is earlier)." ***Late submission will not be considered without extenuating circumstances.*** *Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working. By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.*

1 (i) We consider the Burgers equation in $\mathbb{R}^1$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

for smooth initial data $u(x,0) = u_0(x)$. It is known that all solutions of (1) remain regular for all time. For $t > 0$, we consider a self-similar solution of the following form

$$u(x,t) = \frac{1}{\lambda(t)}U(X),\ X = \frac{x}{\lambda(t)},$$

which becomes singular at $t = t_*$. Here $\lambda(t) = \sqrt{2a(t_* - t)}$ and a, t_* are positive constants. The boundary conditions are $U(X), U'(X) \to 0$ faster than algebraically as $|X| \to \infty$.
If a non-zero U *were* available, that would contradict the above regularity property. In the following steps we will confirm that $U \equiv 0$ is the only possible solution, consistent with the regularity.

(a) By direct computations show that $\dfrac{\partial X}{\partial t} = \dfrac{a}{\lambda(t)^2}X$ and express $\dfrac{\partial u}{\partial t}, \dfrac{\partial u}{\partial x}$ and $\dfrac{\partial^2 u}{\partial x^2}$ in terms of transformed variables, that is, U and its derivatives.

(8 marks)

(b) Using (a) derive the following equation for U from (1)

$$UU' + a(U + XU') = \nu U''. \tag{2}$$

(4 marks)

(c) From (2) derive

$$\frac{U^2}{2} + aXU = \nu U'. \tag{3}$$

(5 marks)

(d) From (3) show that

$$\frac{a}{2}\int_{-\infty}^{\infty} U(X)^2 dX + \nu\int_{-\infty}^{\infty} U'(X)^2 dX = 0,$$

hence prove that the only solution to (3) is the trivial solution, that is, $U \equiv 0$.

(8 marks)

 Question 1 continued on next page

1 (continued)

(ii) Consider a model equation for the vorticity ω in $\mathbb{R}^1$

$$\frac{\partial \omega}{\partial t} = \omega H[\omega], \tag{4}$$

for the initial condition $\omega_0(x) = \dfrac{1}{1+x^2}$. Here $H[\omega]$ denotes the Hilbert transform $H[\omega](x) = \dfrac{1}{\pi} ⨍_{-\infty}^{\infty} \dfrac{\omega(y)}{x-y} dy$, defined with a principal-value integral. We will show, in two different ways, that the solution is just a space-translation at a constant speed.
You may assume the following formulas

$$H\left[\frac{a}{x^2+a^2}\right] = \frac{x}{x^2+a^2}, \; H\left[\frac{x}{x^2+a^2}\right] = -\frac{a}{x^2+a^2},$$

where $a (> 0)$ is a constant.

(a) Assuming that $\omega(x,t) = f(x-ct)$, show that $\partial_t \omega = -cf'(x-ct)$ and $H[\omega](x,t) = H[f](x-ct)$, where c is a constant. ***(5 marks)***

(b) Derive an equation for $f(z)$, where $z = x - ct$. ***(2 marks)***

(c) Find the value of c for which the equation derived in (b) has a solution and hence find the corresponding expression for $\omega(x,t)$. ***(8 marks)***

(d) The exact solution to (4) is given by

$$\omega(x,t) = \frac{4\omega_0(x)}{(2 - tH[\omega_0])^2 + (t\omega_0)^2}.$$

Working from the exact solution, confirm the conclusion in (c).

(10 marks)

2 Consider the steady velocity field governed by the Navier-Stokes equations

$$(u_r, u_\phi, u_z) = (-ar, u(r), 2az),$$

in cylindrical coordinates (r, ϕ, z), where $a (> 0)$ is a constant. We will determine the solution $u(r)$ and the pressure p, using the formulae for the Navier-Stokes equations in cylindrical coordinates given below.

(i) Simplify the equation for the r-component to

$$a^2 r - \frac{u(r)^2}{r} = -\frac{\partial p}{\partial r}.$$

(5 marks)

(ii) Simplify the equation for the z-component to

$$4a^2 z = -\frac{\partial p}{\partial z}.$$

(5 marks)

(iii) Taking the ϕ-component, show that the azimuthal component of velocity $u_\phi = u(r)$ satisfies

$$a\left(r\frac{du}{dr} + u\right) + \nu\left(\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2}\right) = 0.$$

(6 marks)

(iv) Show that $\sigma(r) \equiv ru(r)$ satisfies

$$ar\frac{d\sigma}{dr} + \nu\left(\frac{d^2\sigma}{dr^2} - \frac{1}{r}\frac{d\sigma}{dr}\right) = 0.$$

(12 marks)

(v) By finding a solution $\sigma(r)$, show that the corresponding steady velocity field $u(r)$ is given by

$$u(r) = \frac{C}{r}\left(1 - \exp\left(-\frac{ar^2}{2\nu}\right)\right),$$

where C is a constant. ***(10 marks)***

 Question 2 continued on next page

2 (continued)

(vi) Using (i) and (ii) show that the pressure is given by

$$p = -a^2 \left(\frac{r^2}{2} + 2z^2 \right) + \int_0^r \frac{u(s)^2}{s} ds + C',$$

where C' is a constant. ***(12 marks)***

The Navier-Stokes equations in cylindrical coordinates

$$\begin{cases} \dfrac{\partial u_r}{\partial t} + u_r \dfrac{\partial u_r}{\partial r} + \dfrac{u_\phi}{r} \dfrac{\partial u_r}{\partial \phi} + u_z \dfrac{\partial u_r}{\partial z} - \dfrac{u_\phi^2}{r} = -\dfrac{\partial p}{\partial r} + \nu \left(\triangle u_r - \dfrac{u_r}{r^2} - \dfrac{2}{r^2} \dfrac{\partial u_\phi}{\partial \phi} \right), \\ \dfrac{\partial u_\phi}{\partial t} + u_r \dfrac{\partial u_\phi}{\partial r} + \dfrac{u_\phi}{r} \dfrac{\partial u_\phi}{\partial \phi} + u_z \dfrac{\partial u_\phi}{\partial z} + \dfrac{u_r u_\phi}{r} = -\dfrac{1}{r} \dfrac{\partial p}{\partial \phi} + \nu \left(\triangle u_\phi - \dfrac{u_\phi}{r^2} + \dfrac{2}{r^2} \dfrac{\partial u_r}{\partial \phi} \right), \\ \dfrac{\partial u_z}{\partial t} + u_r \dfrac{\partial u_z}{\partial r} + \dfrac{u_\phi}{r} \dfrac{\partial u_z}{\partial \phi} + u_z \dfrac{\partial u_z}{\partial z} - \dfrac{\partial p}{\partial z} + \nu \triangle u_z, \end{cases}$$

where $\triangle$ denotes the Laplace operator in cylindrical coordinates:

$$\triangle f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2},$$

for any scalar function $f(r, \phi.z)$.

End of Question Paper