Data provided: Formula Sheet

MAS420

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2020–21**

Signal Processing **2.5 hours**

This is an open book exam.
Answer all questions.
You can work on the exam during the 24 hour period starting at 10am (GMT), and you must submit your work within 2.5 hours of accessing the exam paper or by the end of the 24 hour period (whichever is earlier). ***Late submission will not be considered without extenuating circumstances.*** *Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working. By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.*
Total marks 50

1 (i) We define $f(t) = p_{\frac{1}{2}}(t)$ for $-1 \leq t \leq 1$. This signal repeats with a period of 2. Sketch the periodic signal between $-3 \leq t \leq 3$ and find its Fourier series coefficients. ***(8 marks)***

(ii) The periodic signal from part (i) is passed through an ideal low-pass filter, $p_{6\pi}(\omega)$, to yield a signal $g(t)$. Write down the Fourier series for $g(t)$. What is the percentage of power lost of the periodic signal after passing through this filter? ***(6 marks)***

2 A linear shift invariant (LSI) system has the system transfer function $H(\omega) = i\omega/(1+i\omega)$.

(i) What is the impulse response function for this LSI system? ***(2 marks)***

(ii) What is the output of this LSI system in the time domain if the input is $f(t) = p_1(t)$? Sketch the output against time. ***(11 marks)***

(iii) What is the output of this LSI sytem in the time domain if the input is $f(t) = \cos(3t) + \sin(2t) + 1$? ***(5 marks)***

3 (i) Find the Nyquist frequency, in Hz, of the complex signal

$$f(t) = e^{it}\,\mathrm{sinc}(t).$$

(3 marks)

(ii) This signal is sampled at 3/4 of its Nyquist frequency and the samples are used to form a signal $g(t)$ by sinc interpolation. Making use of clear diagrams, find $G(\omega)$ and hence $g(t)$. ***(9 marks)***

(iii) Sampling a signal $f(t)$ in the time domain with spacing T results in the summation,

$$\overline{f}_T(t) = \sum_{n=-\infty}^{\infty} f(t - nT). \qquad (1)$$

Equation (1) is equivalent to the time-domain version of the Poisson summation formula,

$$\overline{f}_T(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} F(n\sigma)\mathrm{e}^{in\sigma t}. \qquad (2)$$

Using (1) show that

$$\overline{f}_T(t) = f(t) * \overline{\delta}_T(t), \qquad (3)$$

then take the Fourier transform of (3) to show that

$$\mathcal{F}\left[\,\overline{f}_T(t)\right] = \sigma \sum_{n=-\infty}^{\infty} F(n\sigma)\delta(\omega - n\sigma). \qquad (4)$$

Finally, take the inverse Fourier transform of (4) to prove (2).

(6 marks)

End of Question Paper

Formula sheet

Function Definitions:

Rectangular pulse:

$$p_a(t) = \begin{cases} 1 & |t| \le a \\ 0 & |t| > a \end{cases}$$

Triangular pulse:

$$q_a(t) = \begin{cases} 1 - |t|/a & |t| \le a \\ 0 & |t| > a \end{cases}$$

Step function:

$$U(t) = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases}$$

Fourier Transform Pairs:

$$\begin{aligned}
p_a(t) &\longleftrightarrow 2a\,\mathrm{sinc}(a\omega) \\
q_a(t) &\longleftrightarrow a\,\mathrm{sinc}^2(a\omega/2) \\
\mathrm{sinc}(at) &\longleftrightarrow \frac{\pi}{a} p_a(\omega) \\
\mathrm{sinc}^2(at) &\longleftrightarrow \frac{\pi}{a} q_{2a}(\omega) \\
\mathrm{e}^{-at}U(t) &\longleftrightarrow \frac{1}{a + i\omega} \\
\delta(t) &\longleftrightarrow 1 \\
\delta(t - t_0) &\longleftrightarrow \mathrm{e}^{-i\omega t_0} \\
1 &\longleftrightarrow 2\pi\delta(\omega) \\
\mathrm{e}^{i\omega_0 t} &\longleftrightarrow 2\pi\delta(\omega - \omega_0) \\
\mathrm{e}^{-t^2/2\sigma^2} &\longleftrightarrow \sigma\sqrt{2\pi}\mathrm{e}^{-\sigma^2\omega^2/2}
\end{aligned}$$

Fourier transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-i\omega t}\,\mathrm{d}t$$

Inverse Fourier transform:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\mathrm{e}^{i\omega t}\,\mathrm{d}\omega$$

Duality theorem: If $f(t) \longleftrightarrow F(\omega)$ then $F(t) \longleftrightarrow 2\pi f(-\omega)$

Scaling: If $f(t) \longleftrightarrow F(\omega)$ then $f(at) \longleftrightarrow \frac{1}{|a|}F(\omega/a)$.

Translation: If $f(t) \longleftrightarrow F(\omega)$ then $f(t - t_0) \longleftrightarrow \mathrm{e}^{-i\omega t_0}F(\omega)$.

Frequency Shift: If $f(t) \longleftrightarrow F(\omega)$ then $\mathrm{e}^{i\omega_0 t}f(t) \longleftrightarrow F(\omega - \omega_0)$

Fourier Series: If $f_T(t)$ is periodic with period T then, with $\sigma = 2\pi/T$, the complex Fourier series is

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{in\sigma t}$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t)\mathrm{e}^{-in\sigma t}\ \mathrm{d}t$$

Likewise, the real Fourier series is

$$f_T(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\sigma t + b_n \sin n\sigma t)$$

where

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f_T(t)\cos n\sigma t\ \mathrm{d}t$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f_T(t)\sin n\sigma t\ \mathrm{d}t$$

Parseval's Theorem: If V is a Hilbert space, $\{\phi_n\}$ is an orthonormal basis for V and $f = \sum_n c_n\phi_n$, then

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Plancherel's Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$\int_{-\infty}^{\infty} f(t)g^*(t)\ \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G^*(\omega)\ \mathrm{d}\omega$$

Energy Theorem: If $f(t) \longleftrightarrow F(\omega)$ then

$$\int_{-\infty}^{\infty} |f(t)|^2\ \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\ \mathrm{d}\omega$$

Convolution Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(s)g(t-s)\ \mathrm{d}s \longleftrightarrow F(\omega)G(\omega)$$

Product Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$f(t)g(t) \longleftrightarrow \frac{1}{2\pi}F(\omega) * G(\omega).$$