The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2020–21**

Analytic Number Theory

This is an open book exam.
Answer all questions.
You can work on the exam during the 24 hour period starting at 10am (GMT), and you must submit your work within 3 hours of accessing the exam paper or by the end of the 24 hour period (whichever is earlier). ***Late submission will not be considered without extenuating circumstances.*** *Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working. By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.*

1 Gauss showed that if -2 is a quadratic residue modulo a prime p then p is congruent to either 1 or 3 modulo 8. Using this fact, employ Euclid's original strategy to prove that there are infinitely many primes of the form $8k + 3$. ***(10 marks)***

2 Prove that the Dirichlet series $D(|\mu|, s)$ associated to the absolute value of the Möbius μ-function satisfies the formal identity

$$D(|\mu|, s) = \frac{\zeta(s)}{\zeta(2s)}$$

where $\zeta(s)$ is the Riemann zeta function. ***(10 marks)***

3 Pick a non-trivial character χ of $(\mathbb{Z}/8\mathbb{Z})^*$ and prove by hand that

$$0 < L(1, \chi) < 2.$$

(4 marks)

4 (i) Let $f : (0, \infty) \to \mathbb{R}$ be a continuously differentiable function. Prove that

$$f(1) + \ldots + f(n) = \frac{f(1) + f(n)}{2} + \int_1^n f(x)dx + \int_1^n f'(x)P_1(x)dx,$$

where $P_1(x) = x - \lfloor x \rfloor - 1/2$ is the first periodical Bernoulli polynomial.

(8 marks)

(ii) Prove that the sequence $a_1, a_2, a_3, \ldots$ where

$$a_n := 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln(n)$$

is convergent. ***(8 marks)***

End of Question Paper