The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2020–2021**

Mathematics Core II **2 hours 30 minutes**

This is an open book exam.

Answer all questions.

You can work on the exam during the 24 hour period starting from 10am (BST), and you must submit your work within 2 hours and 30 minutes of accessing the exam paper or by the end of the 24 hour period (whichever is earlier).

Late submission will not be considered without extenuating circumstances. *Calculations should be performed by hand. A university-approved calculator may be used. The use of any other calculational device, software or service is not permitted. To gain full marks, you will need to show your working.*

By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.

Section A

This section is worth 30 marks. Full working is required to earn all the marks available.

A1 (i) Use row operations to find the value of k for which the following system of equations has solutions:

$$\begin{aligned} x + y + z &= 2 \\ 2x - y + 2z &= 1 \\ 5x - 4y + 5z &= k \end{aligned}$$

For this value of k, give all solutions to the system of equations.

(ii) Evaluate the determinant $\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ 5 & -4 & 5 \end{vmatrix}$, and remark on the significance of this to the solution of the system of equations. ***(5 marks)***

A2 Let $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$.

(i) Compute A^2 explicitly with matrix multiplication.

(ii) Show that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigenvectors of A, and give the corresponding eigenvalues.

(iii) Write down a matrix M such that $M^{-1}AM$ is a diagonal matrix D, which you should also state.

(iv) Show that $M^{-1}A^nM = (M^{-1}AM)^n$, and hence give a formula for A^n. ***(7 marks)***

A3 Consider the surface $z = f(x, y) = x^3 + y^3 - 3x - 3y$.

(i) Give the tangent plane to the surface at $(x, y) = (1, 0)$. What is the normal vector to the surface at that point?

(ii) Determine the stationary points of the surface, and classify them as maximum points, minimum points, or saddle points. ***(5 marks)***

A4 Calculate the length of the graph of the function $f(x) = \frac{4}{3}x^{\frac{3}{2}}$ over the interval $[0, 2]$. ***(4 marks)***

A5 Consider the region $D \subset \mathbb{R}^2$ defined as the set of points (x, y) such that $1 \leq x^2 - y^2 \leq 4$ and $0 \leq y \leq x/2$.

(i) Sketch this region.

(ii) Define $u = x^2 - y^2$ and $v = y/x$. Compute the Jacobian $\dfrac{\partial(u,v)}{\partial(x,y)}$.

(iii) Compute the integral $\displaystyle\iint_D \frac{x^2 - y^2}{x^2}\, dx\, dy$ by changing the variables to u and v. ***(6 marks)***

A6 A Latin square of order n is an $n \times n$ square in which each digit from 1 to n appears exactly once in each row and column. For example, a completed Sudoku grid, as appears in many puzzle sources, is an example of a Latin square of order 9, with an additional constraint on the 3×3-boxes that make up the 9×9-grid.

If we regard a Latin square of order 9 as a 9×9-matrix, show that its determinant is divisible by 405. ***(3 marks)***

Section B

Each question or incomplete statement in this section is followed by four possible options of which exactly one is correct. Mark clearly the correct answer on the question paper. ***(10 marks)***

B1 Let $P = (3, 4, 5)$. What is the distance of P from $(0, 0, 0)$?

A. 5 **B.** 6 **C.** $5\sqrt{2}$ **D.** $6\sqrt{2}$

B2 A is a 2×3-matrix, B is a 2×1-matrix and C is a 4×2-matrix. Which of the following products makes sense?

A. BA^T **B.** CB^T **C.** B^TA **D.** C^TA

B3 What is the inverse matrix of $\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$?

A. $\begin{pmatrix} 1 & -1 \\ 1 & \frac{1}{2} \end{pmatrix}$ **B.** $\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$ **C.** $\begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$ **D.** $\begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$

B4 Which of the following is an eigenvector of $\begin{pmatrix} 12 & 7 \\ 8 & 2 \end{pmatrix}$ with eigenvalue -2?

A. $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ **B.** $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ **C.** $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ **D.** $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

B5 What is the product of the eigenvalues of $\begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}$?

A. 1 **B.** 2 **C.** 3 **D.** 4

B6 Consider the function $f(x,y) = e^{xy^2}$. What is $\dfrac{\partial f}{\partial y}$?

A. e^{xy^2} **B.** $y^2 e^{xy^2}$ **C.** xe^{xy^2} **D.** $2xye^{xy^2}$

B7 The real number x satisfies $\sinh x = 1$. What is $\sinh 2x$?

A. 0 **B.** $2\sqrt{2}$ **C.** 1 **D.** $\sqrt{2}$

B8 One eigenvector of a real symmetric 2×2-matrix is $\begin{pmatrix} 3 & 2 \end{pmatrix}$. What is the other?

A. $\begin{pmatrix} -2 & 3 \end{pmatrix}$ **B.** $\begin{pmatrix} 2 & 3 \end{pmatrix}$ **C.** $\begin{pmatrix} -3 & -2 \end{pmatrix}$ **D.** $\begin{pmatrix} 0 & 0 \end{pmatrix}$

B9 The integral $\int_{-1}^{1} \int_{0}^{y+1} x \, dx \, dy$ can also be computed with

A. $\int_{0}^{2} \int_{x-1}^{1} x \, dy \, dx$ **B.** $\int_{0}^{y+1} \int_{0}^{2} x \, dy \, dx$ **C.** $\int_{0}^{2} \int_{y-1}^{1} x \, dy \, dx$ **D.** $\int_{-1}^{1} \int_{0}^{x+1} x \, dy \, dx$

B10 What is $\int_{-1}^{1} \int_{y}^{1} x \, dx \, dy$?

A. 3 **B.** 4/3 **C.** 1 **D.** 2/3

End of Question Paper